VECTORS

CIE A Level Mathematics 9709 & Further Mathematics 9231

Complete Classroom Notes, Worked Examples, Exam Practice and Solutions

Coverage

9709 Pure Mathematics 3, Section 3.7: vector notation, vector arithmetic, magnitude, unit vectors, position/displacement vectors, lines in 3D, parallel/intersecting/skew lines, scalar product, angles and perpendicular feet.

9231 Further Pure Mathematics 1, Section 1.6: planes in three forms, vector product, line-plane and plane-plane geometry, intersections, perpendicular feet, angles, shortest distance between skew lines and common perpendiculars.

Designed for teaching, revision and examination preparation

Aligned to the Cambridge International 2026–2027 syllabuses.

Contents

1 How to use these notes

These notes are organised in two layers. **Part I** covers the full vector content required for Cambridge International A Level Mathematics (9709). **Part II** extends the same ideas to the Further Mathematics (9231) syllabus. Each subsection contains concise theory, worked examples, exam traps and graded practice. The final sections contain mixed examination-style questions and fully worked solutions.

Exam technique

Cambridge vector questions are usually not difficult because of algebra alone. They test whether you can translate geometry into vector equations. Before calculating, write down what each vector means: a *position vector*, a *direction vector*, a *normal vector*, or a *displacement*.

2 Part I: CIE 9709 Vectors

2.1 Vector notation and basic language

A vector has both **magnitude** and **direction**. A scalar has magnitude only.

Common notations are

$$\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \mathbf{a} = x\mathbf{i} + y\mathbf{j}, \qquad \mathbf{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

If A and B are points, then $\overrightarrow{AB}$ is the displacement from A to B.

Key idea

If the position vectors of A and B are $\mathbf{a}$ and $\mathbf{b}$, then

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}.$$

The order matters: destination minus starting point.

Worked example 1: displacement vector

The points $A(2, -1, 4)$ and $B(7, 3, -2)$ have position vectors

$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix}.$$

Hence

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 5 \\ 4 \\ -6 \end{pmatrix}.$$

Therefore $\overrightarrow{BA} = (-5, -4, 6)^T$.

2.2 Addition, subtraction and scalar multiplication

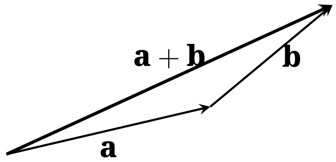
For

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

we have

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}, \qquad k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}.$$

Geometrically, vector addition follows the triangle or parallelogram rule.



Worked example 2: parallelogram geometry

In parallelogram $OABC$, suppose $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. Then the diagonal

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}.$$

Since $\overrightarrow{AB} = \overrightarrow{OC} = \mathbf{c}$, we obtain

$$\boxed{\overrightarrow{OB} = \mathbf{a} + \mathbf{c}}.$$

2.3 Midpoints and simple ratios

The midpoint M of AB has position vector

$$\boxed{\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b})}.$$

More generally, if a point P divides AB internally in the ratio $AP : PB = m : n$, then

$$\mathbf{p} = \frac{n\mathbf{a} + m\mathbf{b}}{m+n}.$$

The general ratio theorem is not explicitly required in 9709, but using it correctly is useful.

Worked example 3: midpoint

A has position vector $(4, -2, 7)^T$ and B has position vector $(-2, 6, 1)^T$. Then

$$\mathbf{m} = \frac{1}{2}\begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

2.4 Magnitude and unit vectors

For $\mathbf{a} = (x, y, z)^T$,

$$\boxed{|\mathbf{a}| = \sqrt{x^2 + y^2 + z^2}}.$$

A unit vector in the direction of non-zero $\mathbf{a}$ is

$$\boxed{\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}}.$$

Worked example 4: unit vector

For $\mathbf{a} = (2, -3, 6)^T$,

$$|\mathbf{a}| = \sqrt{4 + 9 + 36} = 7.$$

Hence the unit vector is

$$\boxed{\frac{1}{7}\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}}.$$

Common error

Do not write a unit vector by dividing by the sum of the components. You must divide by the *magnitude*.

2.5 Position vectors and distance between points

If A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$AB = |\mathbf{b} - \mathbf{a}|.$$

Worked example 5: a 3D distance

For $A(1, 2, -1)$ and $B(4, -2, 5)$,

$$\overrightarrow{AB} = (3, -4, 6)^T,$$

so

$$AB = \sqrt{3^2 + (-4)^2 + 6^2} = \sqrt{61}.$$

2.6 Equation of a straight line

The vector equation of a line is

$$\boxed{\mathbf{r} = \mathbf{a} + t\mathbf{b}}, \qquad t \in \mathbb{R},$$

where $\mathbf{a}$ is the position vector of a fixed point on the line and $\mathbf{b}$ is a non-zero direction vector.

In component form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Thus

$$x = a_1 + tb_1, \quad y = a_2 + tb_2, \quad z = a_3 + tb_3.$$

Worked example 6: line through two points

Find the equation of the line through $A(2, -1, 3)$ and $B(5, 4, 1)$.
A direction vector is

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}.$$

Hence

$$\boxed{\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}}.$$

Any non-zero scalar multiple of $(3,5,-2)^T$ is also a valid direction vector.

Worked example 7: test whether a point lies on a line

Does $P(8,9,-1)$ lie on

$$\mathbf{r}=\begin{pmatrix}2\\-1\\3\end{pmatrix}+t\begin{pmatrix}3\\5\\-2\end{pmatrix}?$$

From x: $2+3t=8 \Rightarrow t=2$. Then $y=-1+5(2)=9$ and $z=3-2(2)=-1$. The same parameter works in all three coordinates, so P lies on the line.

2.7 Parallel, intersecting and skew lines

Consider

$$L_1:\mathbf{r}=\mathbf{a}+\lambda\mathbf{b},\qquad L_2:\mathbf{r}=\mathbf{c}+\mu\mathbf{d}.$$

- The lines are **parallel** if $\mathbf{d}=k\mathbf{b}$ for some non-zero scalar k.
- If direction vectors are not parallel, solve $\mathbf{a}+\lambda\mathbf{b}=\mathbf{c}+\mu\mathbf{d}$.
- If a consistent pair (λ,μ) exists, they **intersect**.
- If no consistent pair exists, they are **skew**.

Worked example 8: intersection of two lines

Let

$$L_1:\mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix},\qquad L_2:\mathbf{r}=\begin{pmatrix}5\\0\\6\end{pmatrix}+\mu\begin{pmatrix}-2\\1\\-3\end{pmatrix}.$$

The direction vectors are opposites, so the lines are parallel. In fact, using $\mu=0$ gives point $(5,0,6)$. On L_1, setting $\lambda=2$ gives $(5,0,6)$. Therefore the two equations represent the **same line**.

Worked example 9: skew lines

$$L_1:\mathbf{r}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix},\qquad L_2:\mathbf{r}=\begin{pmatrix}0\\1\\1\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

From the z-coordinates, $0=1$ would be required for an intersection, which is impossible. The direction vectors are not parallel. Hence the lines are skew.

2.8 The scalar product

For vectors **a** and **b**,

$$\boxed{\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3}$$

and also

$$\boxed{\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta},$$

where θ is the angle between the vectors, $0\le\theta\le\pi$.

Therefore

$$\boxed{\cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}}.$$

Also, $\mathbf{a} \perp \mathbf{b}$ if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Worked example 10: angle between two vectors

Find the angle between $\mathbf{a} = (1, 2, 2)^T$ and $\mathbf{b} = (2, -1, 2)^T$.

$$\mathbf{a} \cdot \mathbf{b} = 2 - 2 + 4 = 4,$$

$$|\mathbf{a}| = 3, \qquad |\mathbf{b}| = 3.$$

Thus

$$\cos\theta = \frac{4}{9}, \qquad \theta = \cos^{-1}\left(\frac{4}{9}\right) \approx 63.6^\circ.$$

2.9 Angle between two lines

The angle between two lines is the acute angle between their direction vectors. If a calculation gives an obtuse angle, replace it by its supplement. Equivalently, use

$$\boxed{\cos\theta = \frac{|\mathbf{b} \cdot \mathbf{d}|}{|\mathbf{b}||\mathbf{d}|}}, \qquad 0 \le \theta \le 90^\circ.$$

Worked example 11: angle between lines

The direction vectors of two lines are $(2, -1, 2)^T$ and $(-1, 2, 2)^T$. Then

$$\mathbf{b} \cdot \mathbf{d} = -2 - 2 + 4 = 0.$$

Hence the lines are perpendicular.

2.10 Foot of the perpendicular from a point to a line

Suppose

$$L : \mathbf{r} = \mathbf{a} + t\mathbf{b}$$

and P has position vector $\mathbf{p}$. Let Q be the foot of the perpendicular on L. Then

$$\mathbf{q} = \mathbf{a} + t\mathbf{b}$$

and the perpendicular condition is

$$(\mathbf{p} - \mathbf{q}) \cdot \mathbf{b} = 0.$$

Solve for t.

Worked example 12: perpendicular foot

Find the foot of the perpendicular from $P(4, 1, 2)$ to

$$L : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

Let

$$Q = (1 + 2t, t, 1 - t).$$

Then

$$\overrightarrow{QP} = \begin{pmatrix} 3 - 2t \\ 1 - t \\ 1 + t \end{pmatrix}.$$

Since QP is perpendicular to the direction vector,

$$(3-2t, 1-t, 1+t) \cdot (2, 1, -1) = 0.$$

So

$$6 - 4t + 1 - t - 1 - t = 0 \Rightarrow 6 - 6t = 0 \Rightarrow t = 1.$$

Hence

$$\boxed{Q = (3, 1, 0)}.$$

2.11 Distance from a point to a line

Once the perpendicular foot Q is known,

$$\boxed{\text{distance} = PQ = |\mathbf{p} - \mathbf{q}|}.$$

Worked example 13: point-to-line distance

Using the previous example,

$$\overrightarrow{QP} = (1, 0, 2)^T,$$

so the distance is

$$\boxed{\sqrt{5}}.$$

2.12 3D geometric applications

Questions may be framed using cuboids, tetrahedra and pyramids. The most reliable method is to assign coordinates to vertices and then translate every geometric statement into vectors.

Worked example 14: cuboid diagonal angle

A cuboid has dimensions $3 \times 4 \times 12$. Let $O = (0, 0, 0)$, $A = (3, 0, 0)$ and $C = (3, 4, 12)$. Find the angle between OA and OC.

$$\overrightarrow{OA} = (3, 0, 0), \qquad \overrightarrow{OC} = (3, 4, 12).$$

Thus

$$\cos\theta = \frac{9}{3\sqrt{9 + 16 + 144}} = \frac{3}{13}.$$

Therefore

$$\boxed{\theta = \cos^{-1}(3/13) \approx 76.7^\circ}.$$

2.13 9709 quick-check practice

Practice

1. Find $\overrightarrow{AB}$ for $A(-1, 3, 2)$ and $B(4, -2, 7)$.
2. Find the midpoint of $P(6, -4, 2)$ and $Q(-2, 8, 10)$.
3. Find a unit vector in the direction $(3, 4, -12)^T$.
4. Find the line through $(1, 2, 3)$ and $(4, -1, 5)$.
5. Determine whether $(7, -4, 7)$ lies on the line in Question 4.
6. Find the angle between $(1, 2, -2)$ and $(2, 0, 1)$.

7. Determine whether the lines

$$\mathbf{r} = (1,0,2) + \lambda(2,1,-1), \quad \mathbf{r} = (5,2,0) + \mu(-4,-2,2)$$

are distinct parallel lines or the same line.
8. Find the point of intersection, if any, of

$$\mathbf{r} = (0,1,2) + \lambda(1,2,1), \quad \mathbf{r} = (3,3,4) + \mu(-1,1,0).$$

9. Find the foot of the perpendicular from $P(2,5,1)$ to $\mathbf{r} = (0,1,0) + t(1,2,2)$.
10. Hence find the shortest distance from P to the line.

3 Part II: CIE 9231 Further Mathematics Vectors

3.1 Planes: three standard forms

A plane can be written in three main forms.

Cartesian form

$$\boxed{ax + by + cz = d}.$$

The vector $\mathbf{n} = (a,b,c)^T$ is normal to the plane.

Normal vector form

$$\boxed{\mathbf{r} \cdot \mathbf{n} = p}.$$

If $\mathbf{n} = (a,b,c)$ and $\mathbf{r} = (x,y,z)$, then this is simply $ax + by + cz = p$.

Parametric vector form

$$\boxed{\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}},$$

where $\mathbf{a}$ is the position vector of a point in the plane and $\mathbf{b}$, $\mathbf{c}$ are two non-parallel directions lying in the plane.

Worked example 15: convert Cartesian to normal form

The plane

$$2x - y + 3z = 7$$

has normal vector $(2,-1,3)^T$. Hence

$$\boxed{\mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 7}.$$

3.2 The vector product

For

$$\mathbf{a} = (a_1, a_2, a_3), \qquad \mathbf{b} = (b_1, b_2, b_3),$$

$$\boxed{\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}}.$$

It is perpendicular to both **a** and **b**.

Also,

$$\boxed{|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta}.$$

Worked example 16: cross product

Let $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (2, -1, 4)$. Then

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 2(4) - 3(-1) \\ 3(2) - 1(4) \\ 1(-1) - 2(2) \end{pmatrix} = \begin{pmatrix} 11 \\ 2 \\ -5 \end{pmatrix}.$$

Check:

$$(11, 2, -5) \cdot (1, 2, 3) = 11 + 4 - 15 = 0.$$

Common error

The cross product is not commutative: $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.

3.3 Finding a plane through three points

If A, B, C are non-collinear, two directions in the plane are $\overrightarrow{AB}$ and $\overrightarrow{AC}$. A normal vector is

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}.$$

Then use $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$.

Worked example 17: plane through three points

Find the plane through $A(1, 0, 2)$, $B(3, 1, 1)$ and $C(0, 2, 4)$.

$$\overrightarrow{AB} = (2, 1, -1), \qquad \overrightarrow{AC} = (-1, 2, 2).$$

Hence

$$\mathbf{n} = (2, 1, -1) \times (-1, 2, 2) = (4, -3, 5).$$

Using point A,

$$4(x - 1) - 3(y - 0) + 5(z - 2) = 0.$$

Thus

$$\boxed{4x - 3y + 5z = 14}.$$

3.4 Converting a parametric plane to Cartesian form

Given

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

a normal vector is $\mathbf{b} \times \mathbf{c}$. Then use the point **a**.

Worked example 18: parametric to Cartesian

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}.$$

A normal is

$$(1, 0, 2) \times (0, 3, -1) = (-6, 1, 3).$$

Hence

$$-6(x-1) + (y-2) + 3(z+1) = 0,$$

so

$$\boxed{-6x + y + 3z = -7}.$$

3.5 Line and plane: parallel, contained or intersecting

Let

$$L : \mathbf{r} = \mathbf{a} + t\mathbf{d}$$

and plane $\Pi : \mathbf{r} \cdot \mathbf{n} = p$.

- If $\mathbf{d} \cdot \mathbf{n} \neq 0$, the line intersects the plane at one point.
- If $\mathbf{d} \cdot \mathbf{n} = 0$, the line is parallel to the plane.
- If it is parallel and a point on the line also satisfies the plane equation, the line lies entirely in the plane.

Worked example 19: line-plane intersection

Find where

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

meets the plane $2x - y + z = 6$.
Substitute $x = 1 + 2t$, $y = t$, $z = 2 - t$:

$$2(1 + 2t) - t + (2 - t) = 6.$$

Thus $4 + 2t = 6$, so $t = 1$. Therefore the intersection point is

$$\boxed{(3, 1, 1)}.$$

3.6 Foot of perpendicular from a point to a plane

For plane

$$\mathbf{r} \cdot \mathbf{n} = p,$$

the perpendicular from point P has direction $\mathbf{n}$:

$$\mathbf{r} = \mathbf{p} + t\mathbf{n}.$$

Substitute into the plane equation to find t.

Worked example 20: perpendicular foot to a plane

Find the foot of the perpendicular from $P(3,1,5)$ to

$$2x - y + 2z = 7.$$

A normal is $\mathbf{n} = (2,-1,2)$. The perpendicular line is

$$(x,y,z) = (3,1,5) + t(2,-1,2).$$

Substitute into the plane:

$$2(3+2t) - (1-t) + 2(5+2t) = 7.$$

So $15 + 9t = 7$, hence $t = -8/9$. Therefore

$$\boxed{Q = \left(\frac{11}{9}, \frac{17}{9}, \frac{29}{9}\right)}.$$

3.7 Distance from a point to a plane

For plane $ax + by + cz = d$ and point (x_0, y_0, z_0),

$$\boxed{D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}}.$$

This formula follows from projection onto the normal vector.

Worked example 21: point-plane distance

Find the distance from $P(1,2,-1)$ to $2x - 2y + z = 4$.

$$D = \frac{|2 - 4 - 1 - 4|}{\sqrt{4+4+1}} = \frac{7}{3}.$$

Thus $\boxed{D = 7/3}$.

3.8 Angle between a line and a plane

Let $\mathbf{d}$ be the line direction and $\mathbf{n}$ the plane normal. If α is the angle between the line and plane, then

$$\boxed{\sin\alpha = \frac{|\mathbf{d}\cdot\mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}}.$$

This is because the angle between $\mathbf{d}$ and $\mathbf{n}$ is $90^\circ - \alpha$.

Worked example 22: line-plane angle

A line has direction $(1,2,2)$ and a plane has normal $(2,-1,2)$. Then

$$\sin\alpha = \frac{|2 - 2 + 4|}{3\cdot 3} = \frac{4}{9}.$$

Hence

$$\boxed{\alpha \approx 26.4^\circ}.$$

3.9 Angle between two planes

The angle between planes equals the acute angle between their normal vectors:

$$\boxed{\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}}.$$

Worked example 23: angle between planes

For

$$\Pi_1 : x + 2y - 2z = 3, \qquad \Pi_2 : 2x - y + 2z = 4,$$

normal vectors are $\mathbf{n}_1 = (1, 2, -2)$ and $\mathbf{n}_2 = (2, -1, 2)$. Their dot product is $2 - 2 - 4 = -4$ and both magnitudes are 3. Hence

$$\cos\theta = \frac{4}{9}, \qquad \boxed{\theta \approx 63.6^\circ}.$$

3.10 Line of intersection of two planes

The direction of the line of intersection is perpendicular to both plane normals, so

$$\boxed{\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2}.$$

Then find one common point by solving the two plane equations, often setting one coordinate conveniently.

Worked example 24: intersection line of two planes

Find the line of intersection of

$$\Pi_1 : x + y + z = 3, \qquad \Pi_2 : 2x - y + z = 1.$$

Normals are $(1, 1, 1)$ and $(2, -1, 1)$, so

$$\mathbf{d} = (1, 1, 1) \times (2, -1, 1) = (2, 1, -3).$$

Set $z = 0$. Then $x + y = 3$ and $2x - y = 1$, giving $x = 4/3$, $y = 5/3$. Thus

$$\boxed{\mathbf{r} = \begin{pmatrix} 4/3 \\ 5/3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}}.$$

3.11 Shortest distance between two skew lines

Let

$$L_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}, \qquad L_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{d}.$$

A vector perpendicular to both directions is

$$\mathbf{n} = \mathbf{b} \times \mathbf{d}.$$

The shortest distance is the magnitude of the projection of $\mathbf{c} - \mathbf{a}$ onto $\mathbf{n}$:

$$\boxed{D = \frac{|(\mathbf{c} - \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{d})|}{|\mathbf{b} \times \mathbf{d}|}}.$$

Worked example 25: shortest distance between skew lines

$$L_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad L_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{b} \times \mathbf{d} = (1, 0, 1) \times (0, 1, 1) = (-1, -1, 1).$$

Also $\mathbf{c} - \mathbf{a} = (0, 2, 1)$. Hence

$$D = \frac{|(0, 2, 1) \cdot (-1, -1, 1)|}{\sqrt{3}} = \frac{|-1|}{\sqrt{3}} = \boxed{\frac{1}{\sqrt{3}}}.$$

3.12 Common perpendicular to two skew lines

The common perpendicular joins a point P on L_1 to a point Q on L_2 such that

$$\overrightarrow{PQ} \cdot \mathbf{b} = 0, \qquad \overrightarrow{PQ} \cdot \mathbf{d} = 0.$$

Write

$$P = \mathbf{a} + \lambda\mathbf{b}, \qquad Q = \mathbf{c} + \mu\mathbf{d},$$

then solve the two scalar equations for λ and μ.

Worked example 26: common perpendicular endpoints

Use the skew lines from Worked example 25.

$$P = (\lambda, 0, \lambda), \qquad Q = (0, 2 + \mu, 1 + \mu).$$

Thus

$$\overrightarrow{PQ} = (-\lambda, 2 + \mu, 1 + \mu - \lambda).$$

Perpendicular to $(1, 0, 1)$ gives

$$-\lambda + 1 + \mu - \lambda = 0 \Rightarrow 1 + \mu - 2\lambda = 0.$$

Perpendicular to $(0, 1, 1)$ gives

$$2 + \mu + 1 + \mu - \lambda = 0 \Rightarrow 3 + 2\mu - \lambda = 0.$$

Solving gives $\lambda = -1/3$, $\mu = -5/3$. Hence

$$P = \left(-\frac{1}{3}, 0, -\frac{1}{3}\right), \qquad Q = \left(0, \frac{1}{3}, -\frac{2}{3}\right).$$

The direction of PQ is proportional to $(-1, -1, 1)$, as expected.

3.13 9231 quick-check practice

Practice

1. Write $3x - 4y + 2z = 9$ in normal vector form.
2. Find the Cartesian equation of the plane through $(1, 0, 0)$ with normal $(2, -1, 3)$.
3. Find $(1, 2, -1) \times (3, 0, 4)$.
4. Find the plane through $A(1, 2, 0)$, $B(2, 0, 1)$ and $C(0, 1, 3)$.

5. Determine whether the line $\mathbf{r} = (1, 0, 2) + t(2, -1, 1)$ is parallel to the plane $x + 2y = 5$.
6. Find the intersection of that line with the plane $2x + y + z = 7$.
7. Find the perpendicular foot from $(3, 2, 1)$ to $x - y + 2z = 4$.
8. Find the angle between line direction $(2, 1, -2)$ and plane $x + 2y + 2z = 6$.
9. Find the angle between planes $x + y + z = 1$ and $x - y + 2z = 3$.
10. Find the line of intersection of $x + y + z = 2$ and $2x - y + z = 1$.
11. Find the shortest distance between

$$\mathbf{r} = (1, 0, 0) + \lambda(1, 1, 0), \quad \mathbf{r} = (0, 1, 1) + \mu(1, -1, 0).$$

12. Find the endpoints of the common perpendicular for the lines in Question 11.

4 Mixed examination-style questions

The following questions are original or carefully adapted to match the demand, structure and skills of Cambridge vector questions. They are suitable for timed classwork and revision. Official Cambridge papers should be used alongside these notes for additional authentic practice.

Set A: 9709 level

A1. Points A, B, C have position vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}.$$

(a) Find $\overrightarrow{AB}$ and AB.

(b) Find the position vector of the midpoint of BC.

(c) Show that angle BAC is acute and find it.

A2. A line passes through $P(2, -1, 4)$ and $Q(6, 3, -2)$.

(a) Find a vector equation of the line.

(b) Determine whether $R(8, 5, -5)$ lies on the line.

(c) Find the point on the line for which $x = 10$.

A3. The lines

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \qquad L_2 : \mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

are given.

(a) Find their point of intersection, if it exists.

(b) Find the acute angle between the lines.

A4. The line L is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

Find the foot of the perpendicular from $P(5,4,1)$ to L, and hence the shortest distance from P to L.

A5. In a cuboid $OABCDEFH$, take $O=(0,0,0)$, $A=(4,0,0)$, $B=(4,3,0)$ and $H=(0,3,12)$. A student calls OB a body diagonal. Explain briefly why this description is incorrect. Then find the acute angle between OH and AB.

Set B: 9231 level

B1. A plane contains $A(1,2,3)$ and has direction vectors $(2,-1,1)$ and $(1,1,0)$.

(a) Find a normal vector.

(b) Find the Cartesian equation of the plane.

B2. The line

$$L : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

and the plane $\Pi : 2x+y-z=5$ are given.

(a) Determine whether L is parallel to Π.

(b) If not, find their point of intersection.

(c) Find the angle between L and Π.

B3. Find the foot of the perpendicular from $P(4,-1,2)$ to the plane $x+2y+2z=3$, and hence find the distance from P to the plane.

B4. Find the acute angle between the planes

$$2x-y+2z=4, \qquad x+2y-2z=1.$$

B5. Find a vector equation for the line of intersection of

$$x+y+z=4, \qquad 2x-y+3z=7.$$

B6. The skew lines

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad L_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

are given.

(a) Find the shortest distance between them.

(b) Find the points $P \in L_1$ and $Q \in L_2$ such that PQ is a common perpendicular.

(c) Write a vector equation for the common perpendicular.

5 Past-paper drill bank

This section is designed to imitate the recurring patterns seen in Cambridge vector questions. Each item is short enough to use as a starter, homework problem or timed drill.

D1. Given $A(2,1,-3)$ and $B(-1,5,2)$, find $\overrightarrow{AB}$ and its magnitude.

D2. Find the unit vector parallel to $(-2,6,3)$ in the opposite direction.

D3. Find k if $(1,k,2)$ is perpendicular to $(2,-1,3)$.

D4. Find the acute angle between $(3,1,-2)$ and $(1,-2,2)$.

D5. Find a line through $(1,-2,4)$ parallel to $(2,3,-1)$.

D6. Find a line through $(1,2,0)$ and $(-2,1,5)$.

D7. Test whether $(7,4,-2)$ lies on $\mathbf{r} = (1,1,1) + t(2,1,-1)$.

D8. Determine whether $\mathbf{r} = (0,1,2) + s(1,2,3)$ and $\mathbf{r} = (2,5,8) + t(2,4,6)$ are the same line.

D9. Find the intersection of $\mathbf{r} = (1,0,0) + s(1,1,1)$ and $\mathbf{r} = (0,2,1) + t(2,-1,1)$.

D10. Find the foot of the perpendicular from $(3,0,4)$ to $\mathbf{r} = (1,1,0) + t(1,-1,2)$.

D11. Find $(2,-1,3) \times (1,4,-2)$.

D12. Find the area of the parallelogram generated by $(1,2,2)$ and $(2,-1,2)$.

D13. Find the plane through $(1,0,2)$ normal to $(3,-2,1)$.

D14. Convert $\mathbf{r} = (1,2,3) + \lambda(1,0,1) + \mu(0,2,-1)$ to Cartesian form.

D15. Find the plane through $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

D16. Determine whether $\mathbf{r} = (1,2,0) + t(2,-1,1)$ lies in $x + 2y = 5$.

D17. Find the line-plane intersection of $\mathbf{r} = (0,1,1) + t(1,2,-1)$ and $x - y + z = 0$.

D18. Find the distance from $(2,1,-1)$ to $2x - y + 2z = 5$.

D19. Find the angle between $\mathbf{d} = (1,2,2)$ and the plane $2x - y + 2z = 7$.

D20. Find the angle between planes $x + y = 2$ and $y + z = 3$.

D21. Find the line of intersection of $x + y + z = 1$ and $x - y + z = 3$.

D22. Find the shortest distance between $\mathbf{r} = s(1,0,1)$ and $\mathbf{r} = (1,2,0) + t(0,1,1)$.

D23. For the lines in D22, find endpoints of a common perpendicular.

D24. Show that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$ for $\mathbf{a} = (1,2,3)$ and $\mathbf{b} = (2,0,-1)$.

6 Fully worked solutions

6.1 Answers to 9709 quick-check practice

1. $\overrightarrow{AB} = (5,-5,5)^T$.
2. $M = (2,2,6)$.
3. Magnitude $= 13$, so unit vector $= (3,4,-12)^T/13$.
4. $\mathbf{r} = (1,2,3)^T + t(3,-3,2)^T$.
5. From x, $t = 2$; then $y = 2 - 6 = -4$, $z = 3 + 4 = 7$, so yes.
6. Dot product $= 0$, so $90°$.
7. Direction vectors are scalar multiples. The point $(5,2,0)$ is reached on the first line at $\lambda = 2$, so the lines are the same.
8. The z-coordinate gives $\lambda = 2$ and the x-coordinate then gives $\mu = 1$, but the y-coordinates are 5 and 4. Hence the equations are inconsistent: the lines are skew and do not intersect.

9. Let $Q = (t, 1 + 2t, 2t)$. Then $(P - Q) \cdot (1, 2, 2) = 0$: $(2 - t) + 2(4 - 2t) + 2(1 - 2t) = 0$, so $12 - 9t = 0$, $t = 4/3$. Hence $Q = (4/3, 11/3, 8/3)$.
10. $PQ = \sqrt{(2/3)^2 + (4/3)^2 + (-5/3)^2} = \sqrt{5}$.

6.2 Answers to 9231 quick-check practice

1. $\mathbf{r} \cdot (3, -4, 2) = 9$.
2. $2(x - 1) - y + 3z = 0$, so $2x - y + 3z = 2$.
3. $(8, -7, -6)$.
4. $\overrightarrow{AB} = (1, -2, 1)$, $\overrightarrow{AC} = (-1, -1, 3)$. A normal is $(-5, -4, -3)$, giving $5x + 4y + 3z = 13$.
5. Plane normal $(1, 2, 0)$; direction dot normal $= 2 - 2 = 0$. Point $(1, 0, 2)$ does not satisfy $x + 2y = 5$, so line is parallel and not contained.
6. Substitute $x = 1 + 2t$, $y = -t$, $z = 2 + t$: $2 + 4t - t + 2 + t = 7$, so $4 + 4t = 7$, $t = 3/4$. Point $(5/2, -3/4, 11/4)$.
7. Normal $(1, -1, 2)$. Perpendicular line $(3, 2, 1) + t(1, -1, 2)$. Substitution gives $3 + t - (2 - t) + 2(1 + 2t) = 4$, so $3 + 6t = 4$, $t = 1/6$. Foot $(19/6, 11/6, 4/3)$.
8. $\sin \alpha = |2 + 2 - 4|/(3 \cdot 3) = 0$, so $\alpha = 0°$: line is parallel to plane.
9. Normals $(1, 1, 1)$ and $(1, -1, 2)$: $\cos \theta = |1 - 1 + 2|/(\sqrt{3}\sqrt{6}) = 2/\sqrt{18} = \sqrt{2}/3$.
10. Direction $(1, 1, 1) \times (2, -1, 1) = (2, 1, -3)$. Set $z = 0$, solve $x + y = 2$, $2x - y = 1$: $(x, y) = (1, 1)$. So $\mathbf{r} = (1, 1, 0) + t(2, 1, -3)$.
11. $\mathbf{b} = (1, 1, 0)$, $\mathbf{d} = (1, -1, 0)$, $\mathbf{b} \times \mathbf{d} = (0, 0, -2)$. Difference of fixed points $= (-1, 1, 1)$. Distance $= |(-1, 1, 1) \cdot (0, 0, -2)|/2 = 1$.
12. Let $P = (1 + \lambda, \lambda, 0)$ and $Q = (\mu, 1 - \mu, 1)$. Then $Q - P = (\mu - 1 - \lambda, 1 - \mu - \lambda, 1)$. Perpendicular to $(1, 1, 0)$ gives $-2\lambda = 0$, so $\lambda = 0$. Perpendicular to $(1, -1, 0)$ gives $2\mu - 2 = 0$, so $\mu = 1$. Thus $P = (1, 0, 0)$, $Q = (1, 0, 1)$.

6.3 Solutions to Set A

A1. $\overrightarrow{AB} = (4, -4, 4)$, so $AB = 4\sqrt{3}$. Midpoint of BC is $(4, 1, 2)$. Also $\overrightarrow{AC} = (2, 2, 2)$ and

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = 8 - 8 + 8 = 8 > 0,$$

so the angle is acute. Since $|AB| = 4\sqrt{3}$, $|AC| = 2\sqrt{3}$,

$$\cos A = \frac{8}{24} = \frac{1}{3}, \qquad A \approx 70.5°.$$

A2. $\overrightarrow{PQ} = (4, 4, -6) = 2(2, 2, -3)$. Hence

$$\mathbf{r} = (2, -1, 4) + t(2, 2, -3).$$

For $R(8, 5, -5)$, $t = 3$ from x, giving $y = 5$, $z = -5$, so R lies on the line. For $x = 10$, $2 + 2t = 10$, $t = 4$, giving $(10, 7, -8)$.

A3. Equating components gives

$$1 + 2\lambda = 5 - \mu, \quad 2 - \lambda = 2\mu, \quad 3 + \lambda = 5.$$

Thus $\lambda = 2$, and from the first $\mu = 0$; the second then gives $0 = 0$. Intersection $(5, 0, 5)$. For the angle,

$$\cos\theta = \frac{|(2, -1, 1) \cdot (-1, 2, 0)|}{\sqrt{6}\sqrt{5}} = \frac{4}{\sqrt{30}},$$

so $\theta \approx 43.1°$.

A4. Let $Q = (1 + 2t, 1 - t, 2 + 2t)$. Then

$$(P - Q) \cdot (2, -1, 2) = 0.$$

That is

$$(4 - 2t, 3 + t, -1 - 2t) \cdot (2, -1, 2) = 0,$$

so $8 - 4t - 3 - t - 2 - 4t = 0$, giving $t = 1/3$. Thus

$$Q = \left(\frac{5}{3}, \frac{2}{3}, \frac{8}{3}\right).$$

Distance

$$PQ = \frac{1}{3}\sqrt{10^2 + 10^2 + (-5)^2} = 5.$$

A5. The statement deliberately requires careful interpretation. OB is a face diagonal, not the body diagonal. $\overrightarrow{OB} = (4, 3, 0)$ and $\overrightarrow{OH} = (0, 3, 12)$. Also $\overrightarrow{AB} = (0, 3, 0)$. Thus the angle between OH and AB satisfies

$$\cos\theta = \frac{9}{(3\sqrt{17})(3)} = \frac{1}{\sqrt{17}},$$

so $\theta \approx 76.0°$.

6.4 Solutions to Set B

B1. A normal is

$$(2, -1, 1) \times (1, 1, 0) = (-1, 1, 3).$$

Using $(1, 2, 3)$,

$$-(x - 1) + (y - 2) + 3(z - 3) = 0,$$

so $\boxed{-x + y + 3z = 10}$.

B2. Plane normal is $(2, 1, -1)$. Direction dot normal is $2 - 2 - 3 = -3 \neq 0$, so the line intersects. Substitution gives

$$2(2 + t) + (1 - 2t) - (3t) = 5 \Rightarrow 5 - 3t = 5 \Rightarrow t = 0.$$

Intersection $(2, 1, 0)$. The angle satisfies

$$\sin\alpha = \frac{3}{\sqrt{14}\sqrt{6}} = \frac{3}{\sqrt{84}},$$

so $\alpha \approx 19.1°$.

B3. Normal $(1, 2, 2)$. Let

$$Q = (4, -1, 2) + t(1, 2, 2).$$

Substitution gives

$$4 + t + 2(-1 + 2t) + 2(2 + 2t) = 3,$$

so $6 + 9t = 3$, $t = -1/3$. Hence

$$Q = \left(\frac{11}{3}, -\frac{5}{3}, \frac{4}{3}\right).$$

Distance $= |t|\,\sqrt{9} = 1$.

B4. Normals are $(2,-1,2)$ and $(1,2,-2)$. Dot product $= 2-2-4=-4$ and both magnitudes are 3, so

$$\theta = \cos^{-1}(4/9) \approx 63.6^\circ.$$

B5. Direction is

$$(1,1,1)\times(2,-1,3)=(4,-1,-3).$$

Set $z=0$: $x+y=4$, $2x-y=7$, so $x=11/3$, $y=1/3$. Thus

$$\boxed{\mathbf{r}=\begin{pmatrix}11/3\\1/3\\0\end{pmatrix}+t\begin{pmatrix}4\\-1\\-3\end{pmatrix}}.$$

B6. Let $\mathbf{b}=(1,1,0)$, $\mathbf{d}=(1,0,1)$ and $\mathbf{c}-\mathbf{a}=(-1,2,2)$. Then

$$\mathbf{b}\times\mathbf{d}=(1,-1,-1).$$

Hence

$$D=\frac{|(-1,2,2)\cdot(1,-1,-1)|}{\sqrt{3}}=\frac{5}{\sqrt{3}}.$$

For endpoints,

$$P=(1+\lambda,\lambda,1),\qquad Q=(\mu,2,3+\mu).$$

Then $Q-P=(\mu-1-\lambda,2-\lambda,2+\mu)$. Perpendicular conditions give

$$\mu+1-2\lambda=0,\qquad 2\mu+1-\lambda=0.$$

Solving gives $\lambda=1/3$, $\mu=-1/3$. Hence

$$P=\left(\frac{4}{3},\frac{1}{3},1\right),\qquad Q=\left(-\frac{1}{3},2,\frac{8}{3}\right).$$

Direction $Q-P=(-5/3,5/3,5/3)$ is proportional to $(-1,1,1)$ and is perpendicular to both line directions. Therefore the common perpendicular may be written

$$\boxed{\mathbf{r}=\begin{pmatrix}4/3\\1/3\\1\end{pmatrix}+s\begin{pmatrix}-1\\1\\1\end{pmatrix}}.$$

7 Compact formula sheet

Concept	Formula / test
Displacement	$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
Magnitude	$\lvert\mathbf{a}\rvert = \sqrt{a_1^2 + a_2^2 + a_3^2}$
Unit vector	$\hat{\mathbf{a}} = \mathbf{a}/\lvert\mathbf{a}\rvert$
Midpoint	$\mathbf{m} = (\mathbf{a} + \mathbf{b})/2$
Line	$\mathbf{r} = \mathbf{a} + t\mathbf{b}$
Dot product	$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
Angle between vectors/lines	$\cos\theta = \lvert\mathbf{a} \cdot \mathbf{b}\rvert/(\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert)$ for acute line angle
Perpendicular vectors	$\mathbf{a} \cdot \mathbf{b} = 0$
Cross product	$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$
Plane	$ax + by + cz = d$ or $\mathbf{r} \cdot \mathbf{n} = p$ or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$
Line parallel to plane	$\mathbf{d} \cdot \mathbf{n} = 0$
Line-plane angle	$\sin\alpha = \lvert\mathbf{d} \cdot \mathbf{n}\rvert/(\lvert\mathbf{d}\rvert\lvert\mathbf{n}\rvert)$
Plane-plane angle	$\cos\theta = \lvert\mathbf{n}_1 \cdot \mathbf{n}_2\rvert/(\lvert\mathbf{n}_1\rvert\lvert\mathbf{n}_2\rvert)$
Point-plane distance	$\lvert ax_0 + by_0 + cz_0 - d\rvert/\sqrt{a^2 + b^2 + c^2}$
Skew-line distance	$\lvert(\mathbf{c} - \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{d})\rvert/\lvert\mathbf{b} \times \mathbf{d}\rvert$

8 Final exam checklist

Before finishing a vector question, check:

- Did you write the correct displacement as *end minus start*?
- Did you use a non-zero direction vector for each line?
- If two lines are parallel, did you check whether they are actually the same line?
- For line intersection, did the same parameter values satisfy all three coordinates?
- For an angle between lines or planes, did you give the acute angle if the question asks for it?
- For a line-plane angle, did you use sine with the plane normal?
- For a perpendicular foot, did you explicitly impose a dot-product-zero condition?
- For a plane through three points, did you use two independent directions and then take a cross product?
- For skew-line distance, did you use a vector perpendicular to both direction vectors?
- Did you state exact values where sensible and round angles consistently?

Source alignment. The scope of these notes follows Cambridge International AS & A Level Mathematics 9709 (2026–2027), Vectors 3.7, and Cambridge International AS & A Level Further Mathematics 9231 (2026–2027), Vectors 1.6. Cambridge's official past-paper pages provide specimen and selected past papers; teachers at registered schools may access additional papers through the School Support Hub. The extended question bank in this handout is original/adapted rather than a verbatim reproduction of copyrighted examination papers.