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Further Pure Mathematics 2

Edexcel AS and A Level Modular Mathematics

FP2

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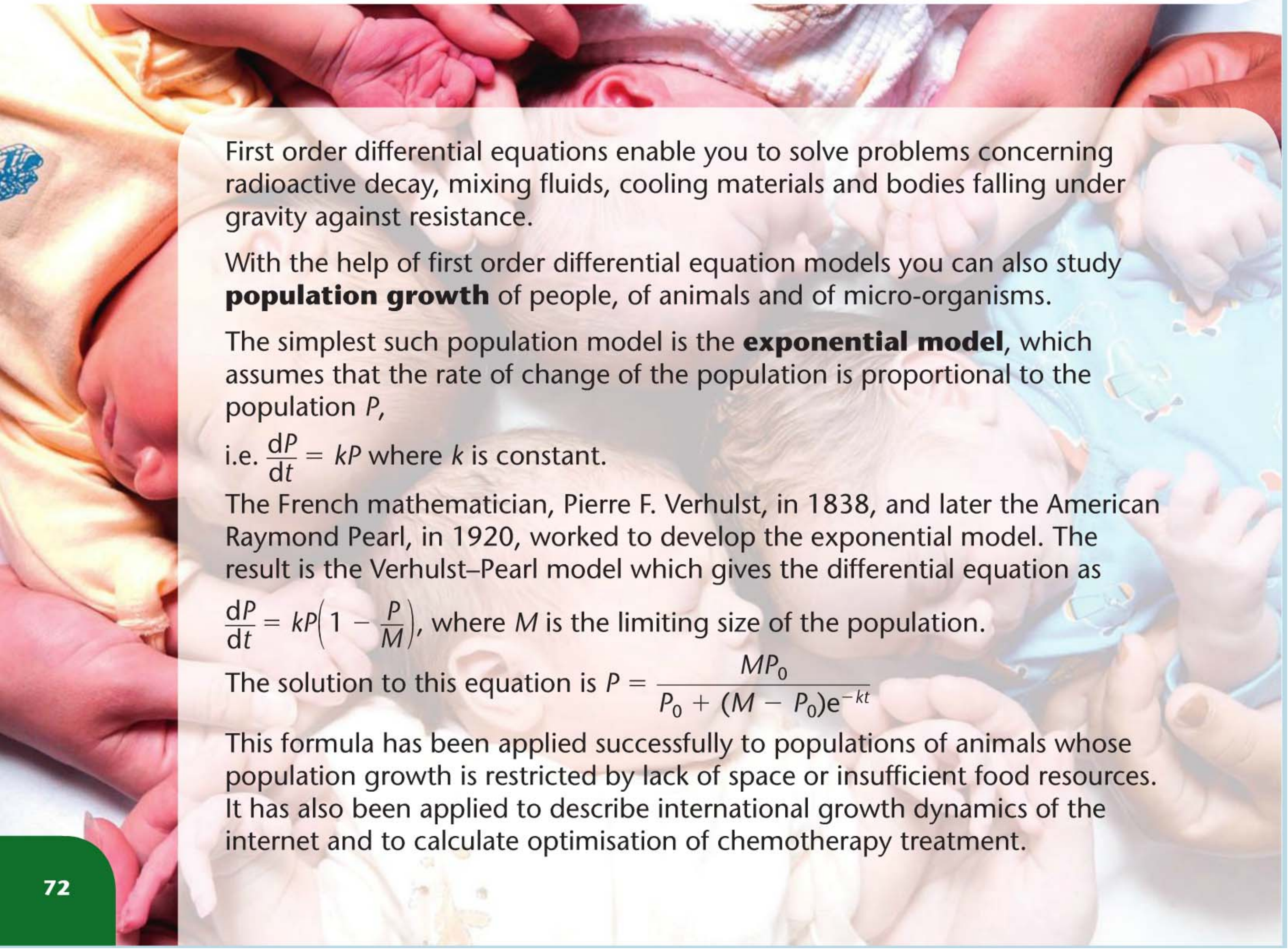
4 First order differential equations

After completing this chapter you will know how to:

- solve first order differential equations by separation of variables and sketch members of the family of solution curves
- solve first order differential equations by the use of an integrating factor
- use a given substitution to transform a differential equation into one that can be solved.



First order differential equations enable you to solve problems concerning radioactive decay, mixing fluids, cooling materials and bodies falling under gravity against resistance.

With the help of first order differential equation models you can also study **population growth** of people, of animals and of micro-organisms.

The simplest such population model is the **exponential model**, which assumes that the rate of change of the population is proportional to the population P,

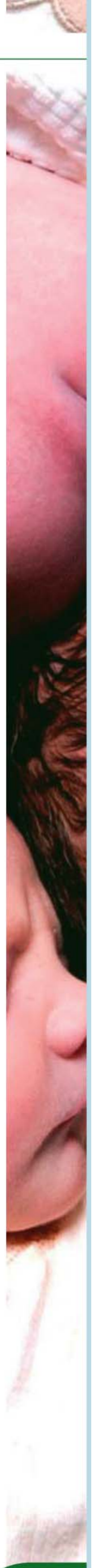
i.e. $\frac{dP}{dt} = kP$ where k is constant.

The French mathematician, Pierre F. Verhulst, in 1838, and later the American Raymond Pearl, in 1920, worked to develop the exponential model. The result is the Verhulst–Pearl model which gives the differential equation as

$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$, where M is the limiting size of the population.

The solution to this equation is $P = \frac{MP_0}{P_0 + (M - P_0)e^{-kt}}$

This formula has been applied successfully to populations of animals whose population growth is restricted by lack of space or insufficient food resources. It has also been applied to describe international growth dynamics of the internet and to calculate optimisation of chemotherapy treatment.



4.1 You can revise both the solution of first order differential equations with separable variables, and the formation of differential equations. You can extend your solution of first order differential equations with separable variables to sketch members of the family of solution curves.

In C4 you met differential equations, and you set up and solved equations using separation of the variables. In this section you will revise the C4 work and learn how to draw a family of solution curves.

- **Remember that when $\frac{dy}{dx} = f(x)g(y)$**

  you can write $\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + C$

Hint: This is called separating the variables and the constant C is the arbitrary constant

When you have integrated and found the general solution, you can let the arbitrary constant take different numerical values, thus generating particular solutions. You can then sketch a graph for each of these solutions. **The curves that are sketched are called a family of solution curves.**

In some questions you will be given a boundary condition, such as $y = 1$ when $x = 0$.

You can use this to find the arbitrary constant. Different boundary conditions will give rise to different particular solutions. The graph of each solution belongs to the family of solution curves.

Example 1

Find the general solution of the differential equation

$$\frac{dy}{dx} = 2$$

and sketch members of the family of solution curves represented by the general solution.

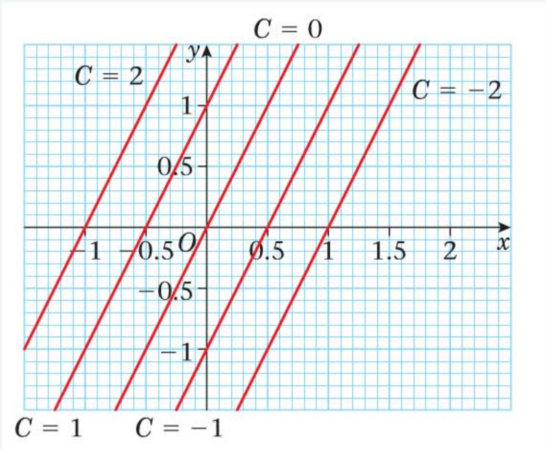
Integrating gives $y = 2x + C$

which is the general solution.

The solution 'curves' corresponding to

$C = -2, -1, 0, 1,$ and 2 are shown below.

This is a straight line equation.



The graphs of $y = 2x + C$ form the family of solution 'curves' for this differential equation.
These are a set of straight lines with gradient 2 and intercept C.

Example 2

Find the general solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

and sketch members of the family of solution curves represented by the general solution.

$\int y\,dy = -\int x\,dx$

Separate the variables and integrate.

$\therefore \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$

which is the general solution.

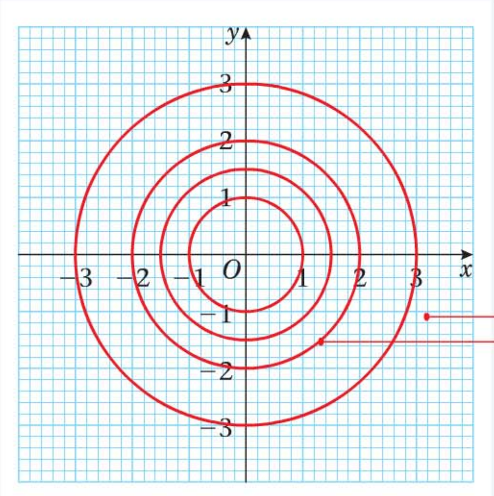
This can be written as $x^2 + y^2 = r^2$,

This is a circle equation.

where $r^2 = 2C$.

The solution curves corresponding to

$C = 0.5,\ 1.125,\ 2,\ 4.5$ are shown below.



The graphs of $x^2 + y^2 = 2C$ form the family of solution curves for this differential equation.

These are a set of circles with centre at the origin and with radius r, where $r^2 = 2C$.

Example 3

Find the general solution of the differential equation

$$\frac{dy}{dx} = -\frac{y}{x}$$

and sketch members of the family of solution curves represented by the general solution.

$\int \frac{1}{y}dy = -\int \frac{1}{x}dx$

Separate the variables and integrate.

$\ln y = -\ln x + C$

$\ln y + \ln x = C$

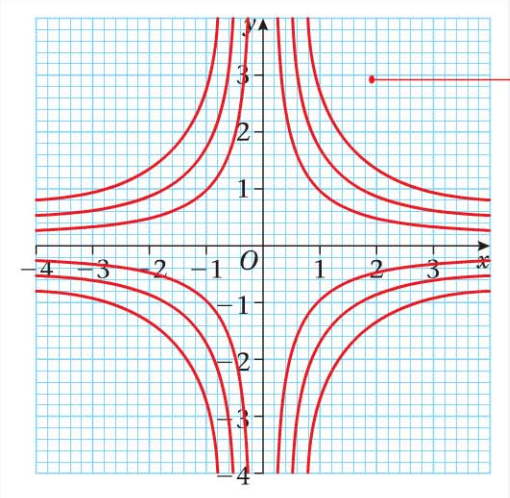
Collect the ln terms together and combine using the laws of logs.

$\ln xy = C$

$|xy| = e^C$

$y = \pm\frac{A}{x}$, where $A = e^C$

The solution curves corresponding to $A = 1, 2, 3$ are shown below



Take exponentials of each side and make y the subject of the formula.

The graphs of $y = \frac{A}{x}$ form the family of solution curves for this differential equation.

These are a set of rectangular hyperbolae with centre at the origin and with the axes as asymptotes.

Example 4

a Find the general solution of the differential equation

$$\frac{dx}{dt} = \sqrt{x}, \qquad t \geqslant 0$$

b Find the particular solutions which satisfy the initial conditions

i $x = 0$ when $t = 0$ **ii** $x = 1$ when $t = 0$

iii $x = 4$ when $t = 0$ **iv** $x = 9$ when $t = 0$

c Sketch the members of the family of solution curves represented by these particular solutions.

a $\int \frac{1}{\sqrt{x}} dx = \int dt$

Separate the variables, which in this question are x and t. t usually denotes time.

$2x^{\frac{1}{2}} = t + c$

$x = \left(\frac{t+c}{2}\right)^2, t \geqslant 0$

Integrate and make x the subject of the formula.

b **i** Substituting $x = 0$ when $t = 0$ gives $c = 0$

$\therefore x = \frac{t^2}{4}, t \geqslant 0$

ii Substituting $x = 1$ when $t = 0$ gives $c = 2$

$\therefore x = \frac{(t+2)^2}{4}, t \geqslant 0$

iii Substituting $x = 4$ when $t = 0$ gives $c = 4$

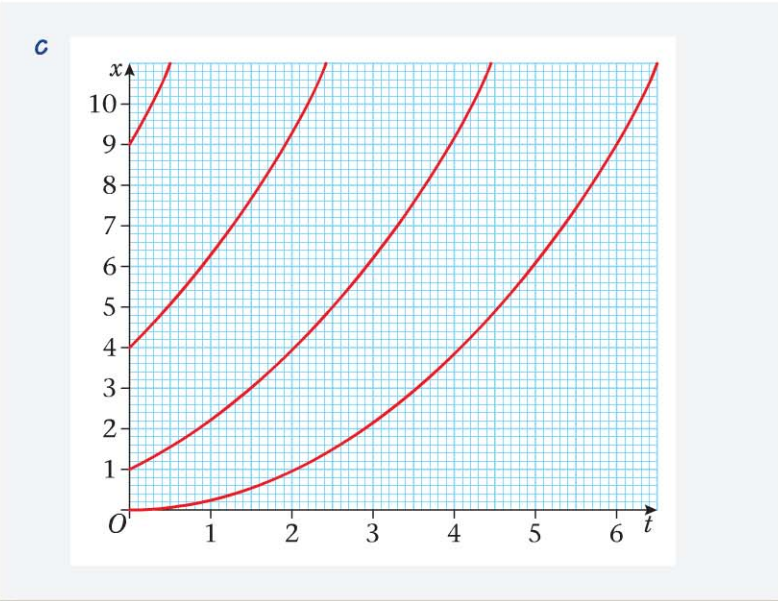
$\therefore x = \frac{(t+4)^2}{4}, t \geqslant 0$

iv Substituting $x = 9$ when $t = 0$ gives $c = 6$

$\therefore x = \frac{(t+6)^2}{4}, t \geqslant 0$

Substitute the initial conditions, i.e. the values of x when $t = 0$, to find c.

Write the equations of the particular solutions.



The graphs of $x = \left(\frac{t + c}{2}\right)^2$, $t \geqslant 0$, form the family of solution curves for this differential equation.

These are parts of **parabolae**.

Exercise 4A

In questions 1–8 find the general solution of the differential equation and sketch the family of solution curves represented by the general solution.

1 $\frac{dy}{dx} = 2x$

2 $\frac{dy}{dx} = y$

3 $\frac{dy}{dx} = x^2$

4 $\frac{dy}{dx} = \frac{1}{x}$, $x > 0$

5 $\frac{dy}{dx} = \frac{2y}{x}$

6 $\frac{dy}{dx} = \frac{x}{y}$

7 $\frac{dy}{dx} = e^y$

8 $\frac{dy}{dx} = \frac{y}{x(x + 1)}$, $x > 0$

9 $\frac{dy}{dx} = \cos x$

10 $\frac{dy}{dx} = y \cot x$, $0 < x < \pi$

11 $\frac{dy}{dt} = \sec^2 t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

12 $\frac{dy}{dx} = x(1 - x)$, $0 < x < 1$

13 Given that a is an arbitrary constant, show that $y^2 = 4ax$ is the general solution of the differential equation $\frac{dy}{dx} = \frac{y}{2x}$.

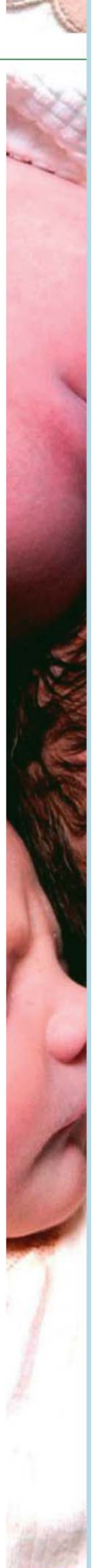
a Sketch the members of the family of solution curves for which $a = \frac{1}{4}$, 1 and 4.

b Find also the particular solution, which passes through the point (1, 3), and add this curve to your diagram of solution curves.

14 Given that k is an arbitrary positive constant, show that $y^2 + kx^2 = 9k$ is the general solution of the differential equation $\frac{dy}{dx} = \frac{-xy}{9 - x^2}$ $\quad |x| \leqslant 3$.

a Find the particular solution, which passes through the point (2, 5).

b Sketch the family of solution curves for $k = \frac{1}{9}, \frac{4}{9}, 1$ and include your particular solution in the diagram.



4.2 You can solve exact equations where one side is the exact derivative of a product and the other side can be integrated with respect to x.

Example 5

Find the general solution of the equation $x^3\frac{dy}{dx} + 3x^2y = \sin x$.

You cannot separate the variables in this example, but you can solve by the method shown.

$x^3\frac{dy}{dx} + 3x^2y = \sin x$

So $\frac{d}{dx}(x^3y) = \sin x$

You can use the product rule $u\frac{dv}{dx} + v\frac{du}{dx} = \frac{d}{dx}(uv)$, with $u = x^3$ and $v = y$, to recognise that $x^3\frac{dy}{dx} + 3x^2y = \frac{d}{dx}(x^3y)$.

$\therefore x^3y = \int \sin x\,dx$

Use integration as the inverse process of differentiation.

$\therefore x^3y = -\cos x + c$

Integrate each side of the equation including an arbitrary constant on the right hand side.

So $y = -\frac{1}{x^3}\cos x + \frac{c}{x^3}$

Make y the subject of the formula by dividing each of the terms on the right hand side by x^3.

In general, note that

- $f(x)\frac{dy}{dx} + f'(x)y = \frac{d}{dx}(f(x)y)$

Use this result to begin to solve an exact differential equation like the one in Example 5.

Exercise 4B

In questions 1–8 find the general solution of the exact differential equation

1 $x\frac{dy}{dx} + y = \cos x$

2 $e^{-x}\frac{dy}{dx} - e^{-x}y = xe^x$

3 $\sin x\frac{dy}{dx} + y\cos x = 3$

4 $\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = e^x$

5 $x^2e^y\frac{dy}{dx} + 2xe^y = x$

6 $4xy\frac{dy}{dx} + 2y^2 = x^2$

7 **a** Find the general solution of the differential equation

$x^2\frac{dy}{dx} + 2xy = 2x + 1.$

b Find the three particular solutions which pass through the points with coordinates $(-\frac{1}{2}, 0)$, $(-\frac{1}{2}, 3)$ and $(-\frac{1}{2}, 19)$ respectively and sketch their solution curves for $x < 0$.

8 **a** Find the general solution of the differential equation

$\ln x\frac{dy}{dx} + \frac{y}{x} = \frac{1}{(x+1)(x+2)}, \qquad x > 1.$

b Find the specific solution which passes through the point (2, 2).

4.3 You can solve first order linear differential equations of the type $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x, by multiplying through the equation by an integrating factor to produce an exact equation.

Example 6

Find the general solution of the equation $\frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^3}$.

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^3}$$

You can multiply this equation by x^3 to make it into an exact equation.

Multiply this equation by x^3

x^3 is called an integrating factor.

$$x^3\frac{dy}{dx} + 3x^2 y = \sin x$$

The solution is $y = -\frac{1}{x^3}\cos x + \frac{c}{x^3}$

This is an exact equation which was solved as Example 5 in the previous section.

Example 7

Solve the general equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x.

Multiply the equation by the integrating factor $f(x)$.

You do this to make the equation exact.

Then $f(x)\frac{dy}{dx} + f(x)\,Py = f(x)Q$ ①

The equation is now exact and so the left hand side is of the form

$$f(x)\frac{dy}{dx} + f'(x)y$$

Compare the left hand side of your differential equation with the format for an exact differential equation.

So $f(x)\frac{dy}{dx} + f(x)Py = f(x)\frac{dy}{dx} + f'(x)y$

$\therefore \quad f'(x) = f(x)P$

Compare the coefficients of y and put them equal.

Dividing by $f(x)$ and integrating

$$\int \frac{f'(x)}{f(x)}\,dx = \int P\,dx$$

This is a ln integral as the numerator is the derivative of the denominator.

$\therefore \quad \ln|f(x)| = \int P\,dx$

$\therefore \quad f(x) = e^{\int P\,dx}$

You need to learn this formula for the integrating factor.

Equation ① becomes

$$e^{\int P\,dx}\frac{dy}{dx} + e^{\int P\,dx}Py = e^{\int P\,dx}Q$$

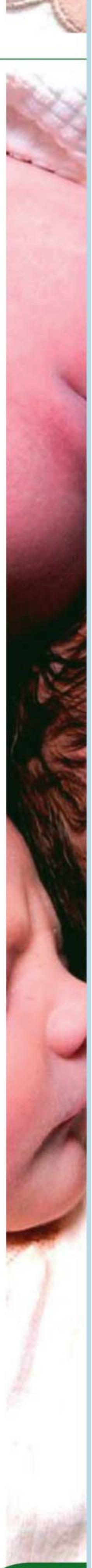
This will lead to a solution provided that these integrals can be found.

$\therefore \quad \frac{d}{dx}\left(e^{\int P\,dx}y\right) = e^{\int P\,dx}Q$

$\therefore \quad e^{\int P\,dx}y = \int e^{\int P\,dx}Q\,dx + C$

The left hand side will always be $y \times$ integrating factor.

This is the solution to the differential equation.



- For the general equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x, you obtain the integrating factor by finding $e^{\int P dx}$
- You obtain the general solution to the differential equation by using $e^{\int P dx} y = \int e^{\int P dx} Q \, dx + C$

Example 8

Find the general solution of the equation $\frac{dy}{dx} - 4y = e^x$.

The integrating factor is $e^{\int P dx} = e^{\int -4 dx} = e^{-4x}$

$\therefore \quad e^{-4x}\frac{dy}{dx} - 4e^{-4x}y = e^x e^{-4x}$

i.e. $\frac{d}{dx}(e^{-4x}y) = e^{-3x}$

$\therefore \quad e^{-4x}y = \int e^{-3x} dx$

So $\quad e^{-4x}y = -\frac{1}{3}e^{-3x} + C$

$y = -\frac{1}{3}e^{x} + Ce^{4x}$

Find the integrating factor.

Multiply the equation by the integrating factor.

Express as the derivative of a product.

Integrate to give the general solution.

Divide every term, including the constant, by the integrating factor to make y the subject of the formula.

Example 9

Find the general solution of the equation $\cos x \frac{dy}{dx} + 2y \sin x = \cos^4 x$

Divide through by $\cos x$

$\therefore \quad \frac{dy}{dx} + 2y \tan x = \cos^3 x. \qquad ①$

The integrating factor is

$e^{\int P dx} = e^{\int 2 \tan x \, dx} = e^{2 \ln \sec x} = e^{\ln \sec^2 x} = \sec^2 x$

$\therefore \quad \sec^2 x \frac{dy}{dx} + 2y \sec^2 x \tan x = \sec^2 x \cos^3 x$

$\therefore \quad \frac{d}{dx}(y \sec^2 x) = \cos x$

$\therefore \quad y \sec^2 x = \int \cos x \, dx$

$\therefore \quad y \sec^2 x = \sin x + c$

$\therefore \quad y = \cos^2 x(\sin x + c)$

Divide by $\cos x$ so that equation is in the correct form i.e. $\frac{dy}{dx} + Py = Q$.

Use properties of ln to simplify the integrating factor.

Multiply equation ① by the integrating factor and simplify the right hand side.

Integrate to give general solution and multiply through by $\cos^2 x$.

Exercise 4C

In questions 1–10 find the general solution of each linear differential equation.

1 $\frac{dy}{dx} + 2y = e^x$

2 $\frac{dy}{dx} + y \cot x = 1$

3 $\frac{dy}{dx} + y \sin x = e^{\cos x}$

4 $\frac{dy}{dx} - y = e^{2x}$

5 $\frac{dy}{dx} + y \tan x = x \cos x$

6 $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$

7 $x^2 \frac{dy}{dx} - xy = \frac{x^3}{x + 2} \quad x > -2$

8 $3x \frac{dy}{dx} + y = x$

9 $(x + 2) \frac{dy}{dx} - y = (x + 2)$

10 $x \frac{dy}{dx} + 4y = \frac{e^x}{x^2}$

11 Find y in terms of x given that
$x \frac{dy}{dx} + 2y = e^x$ and that $y = 1$ when $x = 1$.

12 Solve the differential equation, giving y in terms of x, where
$x^3 \frac{dy}{dx} - x^2 y = 1$ and $y = 1$ at $x = 1$.

13 **a** Find the general solution of the differential equation
$\left(x + \frac{1}{x}\right) \frac{dy}{dx} + 2y = 2(x^2 + 1)^2,$
giving y in terms of x.
b Find the particular solution which satisfies the condition that $y = 1$ at $x = 1$.

14 **a** Find the general solution of the differential equation
$\cos x \frac{dy}{dx} + y = 1, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$
b Find the particular solution which satisfies the condition that $y = 2$ at $x = 0$.

4.4 You can use a given substitution to reduce a differential equation into one of the above types of equation, which you can then solve.

Example 10

Use the substitution $z = \frac{y}{x}$ to transform the differential equation $\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$, $x > 0$, into a differential equation in z and x. By first solving this new equation, find the general solution of the original equation, giving y^2 in terms of x.

Sketch the particular solution curves passing through (0.5, 0), (1, 0), (2, 0) and (3, 0) respectively.

$$z = \frac{y}{x} \rightarrow y = xz \qquad (1)$$

Rearrange the substitution to make y the subject of the formula.

$$\therefore \frac{dy}{dx} = x\frac{dz}{dx} + z \qquad (2)$$

Differentiate to give $\frac{dy}{dx}$ in terms of $\frac{dz}{dx}$.

Substituting into $\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$ gives

$$x\frac{dz}{dy} + z = \frac{x^2 + 3x^2z^2}{2x^2z}$$

$$x\frac{dz}{dx} + z = \frac{x^2(1 + 3z^2)}{2x^2z}$$

Substitute into differential equation using equations (1) and (2).

$$x\frac{dz}{dx} = \frac{(1 + 3z^2)}{2z} - z$$

$$= \frac{1 + z^2}{2z}$$

Rearrange and simplify your equation.

$$\therefore \int \frac{2z}{1 + z^2}\,dz = \int \frac{1}{x}\,dx$$

$$\therefore \ln(1 + z^2) = \ln x + c$$

Separate the variables, then integrate including a constant of integration.

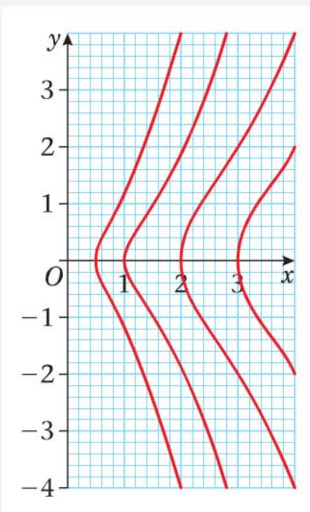
$\therefore (1 + z^2) = Ax$, where A is a positive constant

Take exponentials and let $A = e^c$.

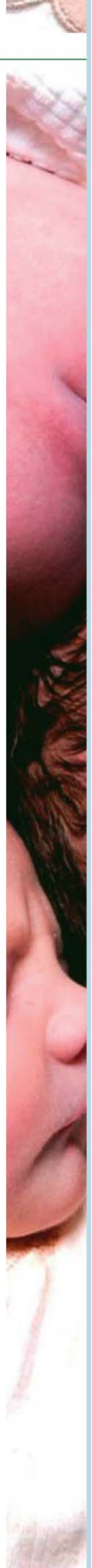
$$\therefore \left(1 + \left(\frac{y^2}{x^2}\right)\right) = Ax$$

Substitute back, to give y in terms of x.

$$\therefore y^2 = x^2(Ax - 1)$$



Use the boundary conditions to obtain $A = 2$, 1, 0.5 and $\frac{1}{3}$. Sketch the curves, as shown.



Example 11

a Use the substitution $z = y^{-1}$ to transform the differential equation $\frac{dy}{dx} + xy = xy^2$, into a differential equation in z and x.

b Solve the new equation, using an integrating factor.

c Find the general solution of the original equation, giving y in terms of x.

a As $z = y^{-1}$, $y = z^{-1}$

$\therefore \frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$

Substituting into $\frac{dy}{dx} + xy = xy^2$ gives

$-\frac{1}{z^2}\frac{dz}{dx} + xz^{-1} = xz^{-2}$

$\therefore \frac{dz}{dx} - xz = -x$

Rearrange the substitution to make y the subject of the formula.

Differentiate to give $\frac{dy}{dx}$ in terms of $\frac{dz}{dx}$.

Rearrange and simplify your equation.

b The integrating factor is $e^{\int -x\,dx} = e^{-\frac{x^2}{2}}$

$\therefore e^{-\frac{x^2}{2}}\frac{dz}{dx} - xe^{-\frac{x^2}{2}}z = -xe^{-\frac{x^2}{2}}$

$\therefore \frac{d}{dx}(e^{-\frac{x^2}{2}}z) = -x\,e^{-\frac{x^2}{2}}$

$\therefore e^{-\frac{x^2}{2}}z = -\int xe^{-\frac{x^2}{2}}\,dx$

$\therefore e^{-\frac{x^2}{2}}z = e^{-\frac{x^2}{2}} + c$

$\therefore z = 1 + ce^{\frac{x^2}{2}}$

Use integrating factor $= e^{\int P\,dx}$.

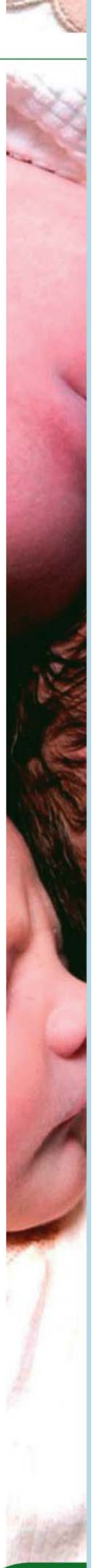
Multiply by integrating factor and establish exact equation.

Integrate to give result then divide each term by integrating factor.

c But as $y = z^{-1}$,

$\therefore y = \frac{1}{1 + ce^{\frac{x^2}{2}}}$

Substitute back to make y the subject of the formula.



Example 12

Use the substitution $u = y - x$ to transform the differential equation

$$\frac{dy}{dx} = \frac{y - x + 2}{y - x + 3}$$

into a differential equation in u and x. By first solving this new equation, show that the general solution of the original equation may be written in the form

$\therefore \quad (y - x)^2 + 6y - 4x - 2c = 0$, where c is an arbitrary constant.

Let $u = y - x$

Then $\frac{du}{dx} = \frac{dy}{dx} - 1$ — Differentiate to give $\frac{du}{dx}$ in terms of $\frac{dy}{dx}$.

Substituting into $\frac{dy}{dx} = \frac{y - x + 2}{y - x + 3}$ gives

$$\frac{du}{dx} + 1 = \frac{u + 2}{u + 3}$$

Make $\frac{dy}{dx}$ the subject of the formula and substitute.

$$\frac{du}{dx} = \frac{u + 2}{u + 3} - 1$$

$$\frac{du}{dx} = \frac{-1}{u + 3}$$

Rearrange and simplify your equation.

$$\int (u + 3)\,du = -\int 1\,dx$$

$$\tfrac{1}{2}u^2 + 3u = -x + c$$

Separate the variables and integrate.

$$\tfrac{1}{2}(y - x)^2 + 3(y - x) = -x + c$$

Substitute back to give your result in terms of x and y.

$$\therefore \quad (y - x)^2 + 6y - 4x - 2c = 0$$

Exercise 4D

In questions 1–4 ,use the substitution $z = \frac{y}{x}$ to transform the given homogeneous differential equation into a differential equation in z and x. By first solving the transformed equation, find the general solution of the original equation, giving y in terms of x.

1 $\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}, \quad x > 0, y > 0$

2 $\frac{dy}{dx} = \frac{y}{x} + \frac{x^2}{y^2}, \quad x > 0$

3 $\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}, \quad x > 0$

4 $\frac{dy}{dx} = \frac{x^3 + 4y^3}{3xy^2}, \; x > 0$

5 Use the substitution $z = y^{-2}$ to transform the differential equation

$$\frac{dy}{dx} + (\tfrac{1}{2}\tan x)\,y = -(2\sec x)y^3, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

into a differential equation in z and x. By first solving the transformed equation, find the general solution of the original equation, giving y in terms of x.

6 Use the substitution $z = x^{\frac{1}{2}}$ to transform the differential equation

$$\frac{dx}{dt} + t^2x = t^2x^{\frac{1}{2}}$$

into a differential equation in z and t. By first solving the transformed equation, find the general solution of the original equation, giving x in terms of t.

7 Use the substitution $z = y^{-1}$ to transform the differential equation

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{(x + 1)^3}{x}y^2$$

into a differential equation in z and x. By first solving the transformed equation, find the general solution of the original equation, giving y in terms of x.

8 Use the substitution $z = y^2$ to transform the differential equation

$$2(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{y}$$

into a differential equation in z and x. By first solving the transformed equation,

a find the general solution of the original equation, giving y in terms of x.

b Find the particular solution for which $y = 2$ when $x = 0$.

9 Show that the substitution $z = y^{-(n-1)}$ transforms the general equation

$$\frac{dy}{dx} + Py = Qy^n,$$

where P and Q are functions of x, into the linear equation $\frac{dz}{dx} - P(n - 1)z = -Q(n - 1)$ (Bernoulli's equation).

10 Use the substitution $u = y + 2x$ to transform the differential equation

$$\frac{dy}{dx} = \frac{-(1 + 2y + 4x)}{1 + y + 2x}$$

into a differential equation in u and x. By first solving this new equation, show that the general solution of the original equation may be written $4x^2 + 4xy + y^2 + 2y + 2x = k$, where k is a constant.

Mixed exercise 4E

1 Solve the equation $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 16}}$ and sketch three solution curves.

2 Solve the equation $\frac{dy}{dx} = xy$ and sketch the solution curves which pass through

a (0, 1) **b** (0, 2) **c** (0, 3).

3 Solve the equation $\frac{dv}{dt} = -g - kv$ given that $v = u$ when $t = 0$, and that u, g and k are positive constants. Sketch the solution curve indicating the velocity which v approaches as t becomes large.

4 Solve the equation $\frac{dy}{dx} + y \tan x = 2 \sec x$

5 Solve the equation $(1 - x^2)\frac{dy}{dx} + xy = 5x \qquad -1 < x < 1$

6 Solve the equation $x\frac{dy}{dx} + x + y = 0$

7 Solve the equation $\frac{dy}{dx} + \frac{y}{x} = \sqrt{x}$

8 Solve the equation $\frac{dy}{dx} + 2xy = x$

9 Solve the equation $x(1 - x^2)\frac{dy}{dx} + (2x^2 - 1)y = 2x^3 \qquad 0 < x < 1$

10 Solve the equation $R\frac{dq}{dt} + \frac{q}{c} = E$ when

a $E = 0$ **b** $E = \text{constant}$ **c** $E = \cos pt$

(R, c and p are constants).

11 Find the general solution of the equation $\frac{dy}{dx} - ay = Q$, where a is a constant, giving your answer in terms of a, when

a $Q = ke^{\lambda x}$ **b** $Q = ke^{ax}$ **c** $Q = kx^n e^{ax}$.

(k, λ and n are constants).

12 Use the substitution $z = y^{-1}$ to transform the differential equation $x\frac{dy}{dx} + y = y^2 \ln x$, into a linear equation. Hence obtain the general solution of the original equation.

13 Use the substitution $z = y^2$ to transform the differential equation $2\cos x\frac{dy}{dx} - y\sin x + y^{-1} = 0$, into a linear equation. Hence obtain the general solution of the original equation.

14 Use the substitution $z = \frac{y}{x}$ to transform the differential equation $(x^2 - y^2)\frac{dy}{dx} - xy = 0$, into a linear equation. Hence obtain the general solution of the original equation.

15 Use the substitution $z = \frac{y}{x}$ to transform the differential equation $\frac{dy}{dx} = \frac{y(x + y)}{x(y - x)}$, into a linear equation. Hence obtain the general solution of the original equation.

16 Use the substitution $z = \frac{y}{x}$ to transform the differential equation $\frac{dy}{dx} = \frac{-3xy}{(y^2 - 3x^2)}$, into a linear equation. Hence obtain the general solution of the original equation.

17 Use the substitution $u = x + y$ to transform the differential equation $\frac{dy}{dx} = (x + y + 1)(x + y - 1)$ into a differential equation in u and x. By first solving this new equation, find the general solution of the original equation, giving y in terms of x.

18 Use the substitution $u = y - x - 2$ to transform the differential equation $\frac{dy}{dx} = (y - x - 2)^2$ into a differential equation in u and x. By first solving this new equation, find the general solution of the original equation, giving y in terms of x.

Summary of key points

- When $\frac{dy}{dx} = f(x)g(y)$ you can write $\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + C$ and by integration you can find the general solution.
- You can assign values to C and sketch a graph for each of these particular solutions. The curves that are sketched are called a family of solution curves. You may use given boundary conditions or initial conditions to find the values for C.
- You can solve exact equations where one side is the exact derivative of a product and the other side can be integrated with respect to x.
- For the general equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x, you obtain the integrating factor by finding $e^{\int P\,dx}$.
  You obtain the general solution to the differential equation by using $e^{\int P\,dx}y = \int e^{\int P\,dx}Q\,dx + C$.
- You may be given a change of variable to transform a given differential equation into a linear equation, which you can then solve.

After completing this chapter you should be able to:

- find general solutions of linear second order differential equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$
- use boundary and initial conditions to find specific solutions
- use a given substitution to transform a second order differential equation into one that can be solved.

5 Second order differential equations

Second order differential equations enable you to solve a variety of problems in economics, physics and engineering. Examples include a tuning fork vibrating in air, the vibration of a spring about an equilibrium position and the variation of charge or of current in an electric circuit.

The shock absorbers on a car or bicycle provide an example of a context in which a spring is incorporated in a system. The motion of the spring is subject to a force proportional to the extension of the spring and also to a damping force, which is assumed to be proportional to the velocity of the vehicle and acts in a direction opposite to the motion.

Newton's Second Law for the motion then gives:

mass $\times$ acceleration = damping force + restoring force

i.e. $m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - cx$

This equation can be written in the form $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$, which is a linear second order equation of the type you will study in this chapter.

The methods used to solve these second order equations were first explained by the mathematicians **Leonhard Paul Euler** (1707–1783) and **Jean le Rond d'Alembert** (1717–1783).

5.1 You can find the general solution of the linear second order differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, where a, b and c are constants and where $b^2 > 4ac$.

In Chapter 4 you solved first order differential equations. You will apply what you have learned, to the solution of second order equations involving the second derivative $\frac{d^2y}{dx^2}$.

Example 1

Solve the equation $a\frac{dy}{dx} + by = 0$, where a and b are constants.

$\int \frac{1}{y}\,dy = -\int \frac{b}{a}\,dx$ — This is a first order equation. Separate the variables.

$\therefore \ln y = -\frac{b}{a}x + \text{constant}$ — Integrate each side.

$\therefore \ln y = -\frac{b}{a}x + \ln A$, where A is constant — Express the constant as a logarithm.

$\therefore y = Ae^{-\frac{b}{a}x}$ — Take exponentials of each side.

The solution of $a\frac{dy}{dx} + by = 0$, where a and b are constants is $y = Ae^{mx}$ where m is a constant.

This suggests that the solution of the second order equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, may also be of the form $y = Ae^{mx}$ where m is a constant.

Example 2

Find the condition for $y = Ae^{mx}$ to be a solution of the equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$.

Let $y = Ae^{mx}$

Then $\frac{dy}{dx} = Ame^{mx}$ and $\frac{d^2y}{dx^2} = Am^2e^{mx}$ — Differentiate with respect to x and differentiate a second time to give the first and second derivative.

Substituting into the differential equation gives

$aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0$

$\therefore Ae^{mx}(am^2 + bm + c) = 0$

So $am^2 + bm + c = 0$ — Factorise, and use $e^{mx} > 0$. This quadratic is called the **auxiliary equation**.

- The equation $am^2 + bm + c = 0$ is called the **auxiliary equation**, and if m is a root of the auxiliary equation then $y = Ae^{mx}$ is a solution of the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$.

The auxiliary equation may have either two real distinct roots, two equal roots or two complex roots.

- When the auxiliary equation has **two real distinct roots** α and β, the general solution of the differential equation is $y = Ae^{\alpha x} + Be^{\beta x}$, where A and B are arbitrary constants.

When you are solving a second order differential equation, the general solution will have **two** arbitrary constants.

Example 3

Find the general solution of the equation $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0$.

Let $y = e^{mx}$, then $\frac{dy}{dx} = me^{mx}$ and $\frac{d^2y}{dx^2} = m^2e^{mx}$

$\therefore\ 2m^2e^{mx} + 5me^{mx} + 3e^{mx} = 0$ — Substitute into the differential equation.

$\therefore\ e^{mx}(2m^2 + 5m + 3) = 0$

As $e^{mx} > 0$, $(2m^2 + 5m + 3) = 0$ — Find the auxiliary equation.

$\therefore\ (2m + 3)(m + 1) = 0$

$\therefore\ m = -\frac{3}{2}$ or $m = -1$ — Solve to give the two values of m.

So the general solution is $y = Ae^{-\frac{3}{2}x} + Be^{-x}$, where A and B are arbitrary constants. — Write the general solution as a sum of multiples of the two independent solutions, using two constants as shown.

Exercise 5A

Find the general solution of each of the following differential equations:

1 $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

2 $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0$

3 $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0$

4 $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 28y = 0$

5 $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0$

6 $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$

7 $3\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 2y = 0$

8 $4\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 2y = 0$

9 $6\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

10 $15\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 2y = 0$

5.2 You can find the general solution of the linear second order differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, where a, b and c are constants and where $b^2 = 4ac$.

Example 4

Show that $y = (A + Bx)e^{3x}$ satisfies the equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$.

Let $y = Ae^{3x} + Bxe^{3x}$, then

$\frac{dy}{dx} = 3Ae^{3x} + 3Bxe^{3x} + Be^{3x}$ and

$\frac{d^2y}{dx^2} = 9Ae^{3x} + 9Bxe^{3x} + 3Be^{3x} + 3Be^{3x}$

$= 9Ae^{3x} + 9Bxe^{3x} + 6Be^{3x}$

So $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 9Ae^{3x} + 9Bxe^{3x} + 6Be^{3x}$

$-6(3Ae^{3x} + 3Bxe^{3x} + Be^{3x})$

$+9(Ae^{3x} + Bxe^{3x}) = 0$

$\therefore$ $y = (A + Bx)e^{3x}$ is a solution of the equation.

Differentiate the expression for y twice and substitute into the differential equation.

Substitute into the left hand side of the differential equation and simplify to show that the result is zero.

- **When the auxiliary equation has two equal roots α, the general solution of the differential equation is**

  $y = (A + Bx)e^{\alpha x}$,

  where A and B are arbitrary constants,

$e^{\alpha x}$ and $xe^{\alpha x}$ are independent solutions of the differential equation when the auxiliary equation has repeated roots.

Example 5

Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$.

Let $y = e^{mx}$, then $\frac{dy}{dx} = me^{mx}$ and $\frac{d^2y}{dx^2} = m^2e^{mx}$

$\therefore$ $m^2e^{mx} + 8me^{mx} + 16e^{mx} = 0$

$\therefore$ $e^{mx}(m^2 + 8m + 16) = 0$

As $e^{mx} > 0$, $(m^2 + 8m + 16) = 0$

$\therefore$ $(m + 4)^2 = 0$

$\therefore$ $m = -4$ only

The general solution in this case is $y = (A + Bx)e^{-4x}$

Find the auxiliary equation as in earlier questions.

Solve the quadratic equation to find the repeated root.

Write the general solution in the form shown above using $\alpha = -4$.

Exercise 5B

Find the general solution of each of the following differential equations:

1 $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$

2 $\frac{d^2y}{dx^2} - 18\frac{dy}{dx} + 81y = 0$

3 $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

4 $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$

5 $\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 0$

6 $16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + y = 0$

7 $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

8 $4\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 25y = 0$

9 $16\frac{d^2y}{dx^2} - 24\frac{dy}{dx} + 9y = 0$

10 $\frac{d^2y}{dx^2} + 2\sqrt{3}\frac{dy}{dx} + 3y = 0$

5.3 You can find the general solution of the linear second order differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, where a, b and c are constants and where $b^2 < 4ac$.

Example 6

Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 16y = 0$

Let $y = e^{mx}$, then $\frac{dy}{dx} = me^{mx}$ and $\frac{d^2y}{dx^2} = m^2e^{mx}$

$\therefore\ m^2e^{mx} + 16e^{mx} = 0$

$\therefore\ e^{mx}(m^2 + 16) = 0$

As $e^{mx} > 0$, $(m^2 + 16) = 0$

$\therefore\ m^2 = -16$ and $m = \pm 4i$

The general solution is $y = Pe^{4ix} + Qe^{-4ix}$,

where P and Q are constants.

This may be written as

$y = P(\cos 4x + i \sin 4x) + Q(\cos 4x - i \sin 4x)$

$= (P + Q) \cos 4x + i(P - Q) \sin 4x$

or $y = A \cos 4x + B \sin 4x$,

where A and B are constants and $A = P + Q$ and $B = i\,(P - Q)$

Find the auxiliary equation and solve to obtain imaginary values for m.

Give the general solution using a similar approach to Example 3 where the roots were real and distinct.

Rewrite $e^{i\theta}$ as $\cos\theta + i \sin\theta$ as you did in section 3.2 of this book.

When the auxiliary equation has **two imaginary roots** $\pm i\omega$, the general solution of the differential equation is

$y = A \cos \omega x + B \sin \omega x$,

where A and B are arbitrary constants.

You may quote this result after finding imaginary roots of the auxiliary equation.

Example 7

Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 34y = 0$.

Let $y = e^{mx}$, then $\frac{dy}{dx} = me^{mx}$ and $\frac{d^2y}{dx^2} = m^2e^{mx}$

$\therefore\ m^2e^{mx} - 6me^{mx} + 34e^{mx} = 0$

$\therefore\ e^{mx}(m^2 - 6m + 34) = 0$

As $e^{mx} > 0$, $\quad m^2 - 6m + 34 = 0$

$\therefore\ m = \frac{6 \pm \sqrt{36 - 136}}{2} = \frac{6 \pm \sqrt{-100}}{2}$

Find the auxiliary equation and solve to obtain conjugate complex values for m.

$\therefore\ m = 3 \pm 5i$

The general solution is $y = Pe^{(3+5i)x} + Qe^{(3-5i)x}$, where P and Q are constants.

Give the general solution using a similar approach to Example 6 where the roots were imaginary.

This may be written as

$y = e^{3x}(Pe^{5ix} + Qe^{-5ix})$

Take out the real factor e^{3x}.

So $y = e^{3x}(A\cos 5x + B\sin 5x)$, where A and B are constants and $A = P + Q$ and $B = i(P - Q)$.

Rewrite $e^{i\theta}$ as $\cos\theta + i\sin\theta$ as you did in Example 6.

- **When the auxiliary equation has two complex roots $p \pm iq$, the general solution of the differential equation is**

  $y = e^{px}(A\cos qx + B\sin qx)$,

  where A and B are arbitrary constants.

You may quote this result after finding conjugate complex roots of the auxiliary equation

Exercise 5C

Find the general solution of each of the following differential equations:

1 $\frac{d^2y}{dx^2} + 25y = 0$

2 $\frac{d^2y}{dx^2} + 81y = 0$

3 $\frac{d^2y}{dx^2} + y = 0$

4 $9\frac{d^2y}{dx^2} + 16y = 0$

5 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

6 $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 17y = 0$

7 $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$

8 $\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 109y = 0$

9 $9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 0$

10 $\frac{d^2y}{dx^2} + \sqrt{3}\frac{dy}{dx} + 3y = 0$

5.4 **You can find the general solution of the linear second order differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, where a, b and c are constants, by using y = complementary function + particular integral.**

- When you are given the equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$,

  you need first to solve $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, as you did in the previous sections.
  The solution, which you obtain, is called the **complementary function** (abbreviated C.F.).

- You then need to find a solution of the equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, where f($x$) will have one of the forms ke^{px}, $A + Bx$, $A + Bx + Cx^2$ or $m \cos \omega x + n \sin \omega x$.

  You find this by using an appropriate substitution and then comparing coefficients.
  The solution is called the **particular integral** (abbreviated P.I.)

Example 8

Find a particular integral of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x)$ when f(x) =

a 3 **b** $2x$ **c** $3x^2$ **d** e^x **e** $13 \sin 3x$.

a Let $y = \lambda$ then $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$

When f(x) = 3, which is constant choose P.I. = λ, also constant.

Substitute into $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 3$

$\therefore 0 - 5 \times 0 + 6\lambda = 3$

Differentiate twice and substitute the derivatives into the differential equation.

$\therefore \lambda = \frac{1}{2}$

Solve equation to give the value of λ.

So a particular integral is $\frac{1}{2}$

b Let $y = \lambda x + \mu$ then $\frac{dy}{dx} = \lambda$ and $\frac{d^2y}{dx^2} = 0$

When f(x) = $2x$, which is a linear function of x choose P.I. = $\lambda x + \mu$, also a linear function.

Substitute into $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2x$

$\therefore 0 - 5 \times \lambda + 6(\lambda x + \mu) = 2x$

As in part **a** differentiate twice and substitute the derivatives into the differential equation.

$\therefore (6\mu - 5\lambda) + 6\lambda x = 2x$

$\therefore 6\mu - 5\lambda = 0$ and $6\lambda = 2$

Equate the constant terms and the coefficients of x to give simultaneous equations, which you can solve to find λ and μ.

$\therefore \lambda = \frac{1}{3}$ and $\mu = \frac{5}{18}$

So a particular integral is $\frac{1}{3}x + \frac{5}{18}$

c Let $y = \lambda x^2 + \mu x + \nu$ then $\frac{dy}{dx} = 2\lambda x + \mu$

and $\frac{d^2y}{dx^2} = 2\lambda$

> As $f(x) = 3x^2$, which is a quadratic function of x let P.I. $= \lambda x^2 + \mu x + \nu$, also a quadratic function.

Substitute into $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 3x^2$

$\therefore 2\lambda - 5(2\lambda x + \mu) + 6(\lambda x^2 + \mu x + \nu) = 3x^2$

> Substitute as you did in parts **a** and **b**.

$\therefore (2\lambda - 5\mu + 6\nu) + (6\mu - 10\lambda)x + 6\lambda x^2 = 3x^2$

$\therefore (2\lambda - 5\mu + 6\nu) = 0$ and $(6\mu - 10\lambda) = 0$ and $6\lambda = 3$

$\therefore \lambda = \frac{1}{2}$ and $\mu = \frac{5}{6}$ and $\nu = \frac{19}{36}$

> Equate the constant terms, the coefficients of x and the coefficients of x^2 to give simultaneous equations, which you can solve to find λ, μ and ν.

So a particular integral is $\frac{1}{2}x^2 + \frac{5}{6}x + \frac{19}{36}$.

d Let $y = \lambda e^x$ then $\frac{dy}{dx} = \lambda e^x$ and $\frac{d^2y}{dx^2} = \lambda e^x$

> As $f(x) = e^x$, which is an exponential function of x let P.I. $= \lambda e^x$, also an exponential function.

Substitute into $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x$

$\therefore \lambda e^x - 5\lambda e^x + 6\lambda e^x = e^x$

$\therefore 2\lambda e^x = e^x$

$\therefore \lambda = \frac{1}{2}$

> Differentiate, substitute and solve to find the value of λ.

So a particular integral is $\frac{1}{2}e^x$

e Let $y = \lambda \sin 3x + \mu \cos 3x$

> As $f(x) = 13 \sin 3x$, which is a trigonometric function of x let P.I. $= \lambda \sin 3x + \mu \cos 3x$, also a similar trigonometric function.

then $\frac{dy}{dx} = 3\lambda \cos 3x - 3\mu \sin 3x$

and $\frac{d^2y}{dx^2} = -9\lambda \sin 3x - 9\mu \cos 3x$

Substitute into $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 13 \sin 3x$

$\therefore -9\lambda \sin 3x - 9\mu \cos 3x$
$\quad -5(3\lambda \cos 3x - 3\mu \sin 3x)$
$\quad +6(\lambda \sin 3x + \mu \cos 3x) = 13 \sin 3x$

$\therefore (-9\lambda + 15\mu + 6\lambda) \sin 3x +$
$(-9\mu - 15\lambda + 6\mu) \cos 3x = 13 \sin 3x$

$\therefore -9\lambda + 15\mu + 6\lambda = 13$ and $-9\mu - 15\lambda + 6\mu = 0$

> Equate coefficients of $\sin 3x$ and of $\cos 3x$ and solve simultaneous equations.

$\therefore \lambda = -\frac{1}{6}$ and $\mu = \frac{5}{6}$

So a particular integral is $-\frac{1}{6} \sin 3x + \frac{5}{6} \cos 3x$

You can find the general solution of the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

by using: $y =$ **complementary function** $+$ **particular integral**.

Example 9

Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x)$ when $f(x) =$

a 3 **b** $2x$ **c** $3x^2$ **d** e^x **e** $13 \sin 3x$

First solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Let $y = e^{mx}$, then $\frac{dy}{dx} + me^{mx}$ and $\frac{d^2y}{dx^2} = m^2e^{mx}$

$\therefore e^{mx}(m^2 - 5m + 6) = 0$

As $e^{mx} > 0$, then $m = 3$ or $m = 2$

So the complementary function is $y = Ae^{3x} + Be^{2x}$, where A and B are arbitrary constants.

The particular integrals were found in Example 8 and so the general solutions are

a $y = Ae^{3x} + Be^{2x} + \frac{1}{2}$,

b $y = Ae^{3x} + Be^{2x} + \frac{1}{3}x + \frac{5}{18}$,

c $y = Ae^{3x} + Be^{2x} + \frac{1}{2}x^2 + \frac{5}{6}x + \frac{19}{36}$,

d $y = Ae^{3x} + Be^{2x} + \frac{1}{2}e^x$,

e $y = Ae^{3x} + Be^{2x} - \frac{1}{6}\sin 3x + \frac{5}{6}\cos 3x$.

First find the complementary function by setting the right hand side of the differential equation equal to zero and solving the resulting equation.

Then use general solution = complementary function + particular integral.

Example 10

Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$

As in Example 9, the complementary function is $y = Ae^{3x} + Be^{2x}$.

The particular integral **cannot be** λe^{2x}, as this is part of the complementary function (sometimes abbreviated c.f.).

So let $y = \lambda x e^{2x}$ then $\frac{dy}{dx} = 2\lambda x e^{2x} + \lambda e^{2x}$

and $\frac{d^2y}{dx^2} = 4\lambda x e^{2x} + 2\lambda e^{2x} + 2\lambda e^{2x} = 4\lambda x e^{2x} + 4\lambda e^{2x}$

Substitute into $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$

$\therefore 4\lambda x e^{2x} + 4\lambda e^{2x} - 5(2\lambda x e^{2x} + \lambda e^{2x}) + 6\lambda x e^{2x} = e^{2x}$

$\therefore -\lambda e^{2x} = e^{2x}$

$\therefore \lambda = -1$

So a particular integral is $-xe^{2x}$

The general solution is $y = Ae^{3x} + Be^{2x} - xe^{2x}$.

The function λe^{2x} is **part of the c.f.** and satisfies the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$, so it **cannot** also satisfy $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$.

Let the P.I be $\lambda x e^{2x}$ and differentiate, substitute and solve to find λ.

Then use general solution = complementary function + particular integral.

Example 11

Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 3$

First consider the equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$

Let $y = e^{mx}$,

then $\frac{dy}{dx} = me^{mx}$ and $\frac{d^2y}{dx^2} = m^2e^{mx}$

$\therefore\ e^{mx}(m^2 - 2m) = 0$

As $e^{mx} > 0$, $\quad m(m - 2) = 0$

$\therefore\ m = 0$ or $m = 2$

So, the complementary function is

$y = A + Be^{2x}$.

The particular integral **cannot be** λ, as this is part of the complementary function.

So let $y = \lambda x$

then $\frac{dy}{dx} = \lambda$ and $\frac{d^2y}{dx^2} = 0$

Substitute into $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 3$

$\therefore\ 0 - 2\lambda = 3$

$\therefore\ \lambda = -1\frac{1}{2}$

So a particular integral is $-\frac{3}{2}x$

The general solution is $y = A + Be^{2x} - 1\frac{1}{2}x$.

Find the complementary function by putting the right hand side of the differential equation equal to zero, and solving the new equation.

Then try to find a particular integral. The right hand side of the original equation was 3, which was a constant and usually this would imply a constant P.I.

As the complementary function includes a constant term 'A', the P.I. cannot also be constant. A value of λ would satisfy $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$ rather than $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 3$.

Multiply the 'expected' particular integral by x and try λx instead.

Then use general solution = complementary function + particular integral.

You may find this table helpful when trying particular integrals.

Form of f(x)	Form of particular integral
k	λ
kx	$\lambda + \mu x$
kx^2	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m \cos \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$n \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \cos \omega x + n \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$

You should **learn these** particular integrals. When the P.I. is non standard the question will probably suggest the form of P.I. to you. (See question 11 in Exercise 5D.)

Exercise 5D

In questions 1–10 solve each of the differential equations, giving the general solution.

1 $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 10$

2 $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x$

3 $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 12e^{2x}$

4 $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 5$

5 $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 8x + 12$

6 $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 25\cos 2x$

7 $\frac{d^2y}{dx^2} + 81y = 15e^{3x}$

8 $\frac{d^2y}{dx^2} + 4y = \sin x$

9 $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 25x^2 - 7$

10 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 26y = e^x$

11 **a** Find the value of λ for which $\lambda x^2 e^x$ is a particular integral for the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$$

b Hence find the general solution.

5.5 **You can use boundary conditions, to find a specific solution of the linear second order differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, where a, b and c are constants, or initial conditions to find a specific solution of the linear second order differential equation $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$, where a, b and c are constants.**

Example 12

Find y in terms of x, given that $\frac{d^2y}{dx^2} - y = 2e^x$, and that $\frac{dy}{dx} = 0$ and $y = 0$ at $x = 0$.

First consider the equation $\frac{d^2y}{dx^2} - y = 0$

Let $y = e^{mx}$, then $\frac{dy}{dx} = me^{mx}$ and $\frac{d^2y}{dx^2} = m^2e^{mx}$

Find the complementary function by putting the right hand side of the differential equation equal to zero, and solving the new equation.

$\therefore\ m^2e^{mx} - e^{mx} = 0$

$\therefore\ e^{mx}(m^2 - 1) = 0$

As $e^{mx} > 0$, $(m + 1)(m - 1) = 0$

$\therefore\ m = \pm 1$

So, the **complementary function** is
$y = Ae^x + Be^{-x}$.

The **particular integral cannot** be λe^x,

as this is part of the complementary function.

> As λe^x satisfies $\frac{d^2y}{dx^2} - y = 0$, it cannot also satisfy $\frac{d^2y}{dx^2} - y = 2e^x$.

So let $y = \lambda x e^x$

then $\frac{dy}{dx} = \lambda x e^x + \lambda e^x$ and $\frac{d^2y}{dx^2} = \lambda x e^x + \lambda e^x + \lambda e^x$

> Instead try $\lambda x e^x$ as a P.I.

Substitute into $\frac{d^2y}{dx^2} - y = 2e^x$

$\therefore\ \lambda x e^x + \lambda e^x + \lambda e^x - \lambda x e^x = 2e^x$

$\therefore\ \lambda = 1$

So a **particular integral** is xe^x

The **general solution** is

$y = Ae^x + Be^{-x} + xe^x$.

Since $y = 0$ at $x = 0$: $0 = A + B$

$\Rightarrow\quad A + B = 0$

> Substitute the boundary condition, $y = 0$ at $x = 0$, into the general solution to obtain an equation relating A and B.

Differentiating $y = Ae^x + Be^{-x} + xe^x$ with respect to x

gives $\frac{dy}{dx} = Ae^x - Be^{-x} + e^x + xe^x$

Since $\frac{dy}{dx} = 0$ at $x = 0$: $0 = A - B + 1$

$\Rightarrow\quad A - B = -1$

> Substitute the second boundary condition, $\frac{dy}{dx} = 0$ at $x = 0$, into the derivative of the general solution, to obtain a second equation relating A and B.

Solving the simultaneous equations gives

$A = -\frac{1}{2}$, and $B = \frac{1}{2}$.

> Solve the two equations to find values for A and B.

And so $y = -\frac{1}{2}e^x + \frac{1}{2}e^{-x} + xe^x$

is the required solution.

Example 13

Given that a particular integral is of the form $\lambda \sin 2t$, find the solution of the differential equation $\frac{d^2x}{dt^2} + x = 3 \sin 2t$, for which $x = 0$ and $\frac{dx}{dt} = 1$ when $t = 0$.

First consider the equation $\frac{d^2x}{dt^2} + x = 0$

Let $x = e^{mt}$, then $\frac{dx}{dt} = me^{mt}$ and $\frac{d^2x}{dt^2} = m^2e^{mt}$

> Again find the **complementary function** by putting the right hand side of the differential equation equal to zero, and solving the new equation.

$\therefore\ m^2e^{mt} + e^{mt} = 0$

$\therefore\ e^{mt}(m^2 + 1) = 0$

As $e^{mt} > 0$, $\quad m^2 = -1 \quad \therefore\ m = \pm i$

So, the **complementary function** is

$x = A \cos t + B \sin t$.

The **particular integral is** $\lambda \sin 2t$,

So let $x = \lambda \sin 2t$

You are told that the P.I. is $\lambda \sin 2t$ in the question, so use this in your solution.

then $\frac{dx}{dt} = 2\lambda \cos 2t$

and $\frac{d^2x}{dt^2} = -4\lambda \sin 2t$

Substitute into $\frac{d^2x}{dt^2} + x = 3 \sin 2t$

$\therefore\ -4\lambda \sin 2t + \lambda \sin 2t = 3 \sin 2t$

$\therefore\ \lambda = -1$

So a **particular integral** is $-\sin 2t$

The **general solution** is

$x = A \cos t + B \sin t - \sin 2t$

Then use general solution = complementary function + particular integral.

Since $x = 0$ at $t = 0$: $0 = A$

$\Rightarrow\ A = 0$

Substitute the initial condition, $x = 0$ at $t = 0$, into the general solution to obtain $A = 0$.

Differentiating $x = B \sin t - \sin 2t$ with respect to t

gives $\frac{dx}{dt} = +B \cos t - 2 \cos 2t$

Since $\frac{dx}{dt} = 1$ at $t = 0$: $1 = B - 2$

Substitute the second initial condition, $\frac{dx}{dt} = 1$ at $t = 0$, into the derivative of the general solution, to obtain a second equation leading to $B = 3$.

$\Rightarrow\ B = 3$

And so $x = 3 \sin t - \sin 2t$ is the required solution.

Exercise 5E

In questions 1–5 find the solution subject to the given boundary conditions for each of the following differential equations.

1 $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12e^x$ $\qquad y = 1$ and $\frac{dy}{dx} = 0$ at $x = 0$

2 $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 12e^{2x}$ $\qquad y = 2$ and $\frac{dy}{dx} = 6$ at $x = 0$

3 $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 42y = 14$ $\qquad y = 0$ and $\frac{dy}{dx} = \frac{1}{6}$ at $x = 0$

4 $\frac{d^2y}{dx^2} + 9y = 16 \sin x$ $\qquad y = 1$ and $\frac{dy}{dx} = 8$ at $x = 0$

5 $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = \sin x + 4 \cos x$ $\qquad y = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$

In questions 6–10 find the solution subject to the given initial conditions for each of the following differential equations.

6 $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2t - 3$ $\qquad x = 2$ and $\frac{dx}{dt} = 4$ when $t = 0$

7 $\frac{d^2x}{dt^2} - 9x = 10 \sin t$ $\qquad x = 2$ and $\frac{dx}{dt} = -1$ when $t = 0$

8 $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 3te^{2t}$ $\qquad x = 0$ and $\frac{dx}{dt} = 1$ when $t = 0$

Hint: let P.I. $= \lambda t^3 e^{2t}$

9 $25\frac{d^2x}{dt^2} + 36x = 18$ $\qquad x = 1$ and $\frac{dx}{dt} = 0.6$ when $t = 0$

10 $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 2t^2$ $\qquad x = 1$ and $\frac{dx}{dt} = 3$ when $t = 0$

5.6 You can use a given substitution to transform a second order differential equation into one of the above types of equation, which you can then solve.

Example 14

Given that $x = e^u$, where u is a function of x, show that

a $x\frac{dy}{dx} = \frac{dy}{du}$ $\qquad$ **b** $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$.

c Hence find the general solution of the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

a As $x = e^u$, $\frac{dx}{du} = e^u = x$

From the chain rule $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = e^u \frac{dy}{dx}$

$= x\frac{dy}{dx}$, as required

Use the chain rule to express $\frac{dy}{dx}$ in terms of $\frac{dy}{du}$.

b $\frac{d^2y}{du^2} = \frac{d}{du}\left(\frac{dy}{du}\right)$

$= \frac{d}{du}\left(e^u\frac{dy}{dx}\right)$

Differentiate the product using the product rule.

$= e^u\frac{dy}{dx} + e^u\frac{d^2y}{dx^2}\frac{dx}{du}$

$= \frac{dy}{du} + x^2\frac{d^2y}{dx^2}$, as $\frac{dx}{du} = e^u = x$

You use the chain rule to differentiate $\frac{dy}{dx}$ with respect to u, by differentiating with respect to x, giving $\frac{d^2y}{dx^2}$, and then multiplying by $\frac{dx}{du}$.

So $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ as required.

c Substitute the results of **a** and **b** into the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

to obtain $\frac{d^2y}{du^2} - \frac{dy}{du} + \frac{dy}{du} + y = 0$

i.e. $\frac{d^2y}{du^2} + y = 0$

This is a second order differential equation with constant coefficients which you know how to solve.

Let $y = e^{mu}$, then $\frac{dy}{du} = me^{mu}$ and $\frac{d^2y}{du^2} = m^2e^{mu}$

$\therefore\ m^2e^{mu} + e^{mu} = 0$

$\therefore\ e^{mu}(m^2 + 1) = 0$

As $e^{mu} > 0$, $m^2 = -1$ $\therefore\ m = \pm i$

So, the **general solution** of the differential equation

$\frac{d^2y}{du^2} + y = 0$ is $y = A\cos u + B\sin u$,

Solve the linear differential equation to give y in terms of u.

where A and B are arbitrary constants.

$x = e^u \Rightarrow u = \ln x$ and the general solution of the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ is

Then use $u = \ln x$ to give y in terms of x.

$y = A\cos(\ln x) + B\sin(\ln x)$

Exercise 5F

In questions 1–6 find the general solution of each differential equation using the substitution $x = e^u$, where u is a function of x.

1 $x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 4y = 0$

2 $x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0$

3 $x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y = 0$

4 $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 28y = 0$

5 $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - 14y = 0$

6 $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 2y = 0$

7 Use the substitution $y = \frac{z}{x}$ to transform the differential equation

$x\frac{d^2y}{dx^2} + (2 - 4x)\frac{dy}{dx} - 4y = 0$ into the equation $\frac{d^2z}{dx^2} - 4\frac{dz}{dx} = 0$.

Hence solve the equation $x\frac{d^2y}{dx^2} + (2 - 4x)\frac{dy}{dx} - 4y = 0$, giving y in terms of x.

8 Use the substitution $y = \frac{z}{x^2}$ to transform the differential equation

$x^2\frac{d^2y}{dx^2} + 2x(x + 2)\frac{dy}{dx} + 2(x + 1)^2 y = e^{-x}$ into the equation $\frac{d^2z}{dx^2} + 2\frac{dz}{dx} + 2z = e^{-x}$.

Hence solve the equation $x^2\frac{d^2y}{dx^2} + 2x(x + 2)\frac{dy}{dx} + 2(x + 1)^2 y = e^{-x}$, giving y in terms of x.

9 Use the substitution $z = \sin x$ to transform the differential equation

$\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$ into the equation $\frac{d^2y}{dz^2} - 2y = 2(1 - z^2)$.

Hence solve the equation $\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$, giving y in terms of x.

Mixed exercise 5G

1 Find the general solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$ *E*

2 Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 36y = 0$

3 Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0$

4 Find y in terms of k and x, given that $\frac{d^2y}{dx^2} + k^2y = 0$ where k is a constant, and $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$. *E*

5 Find the solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$ for which $y = 0$ and $\frac{dy}{dx} = 3$ at $x = 0$.

6 Given that the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = e^{2x}$ has a particular integral of the form ke^{2x}, determine the value of the constant k and find the general solution of the equation. *E*

7 Given that the differential equation $\frac{d^2y}{dx^2} - y = 4e^x$ has a particular integral of the form kxe^x, determine the value of the constant k and find the general solution of the equation.

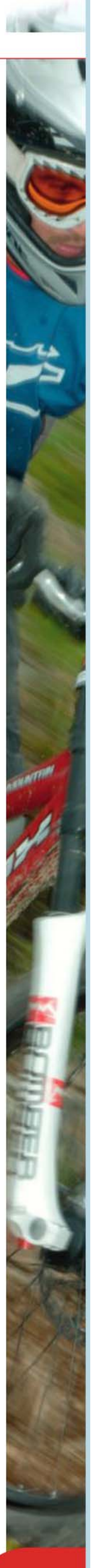
8 The differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4e^{2x}$ is to be solved.

a Find the complementary function.

b Explain why **neither** λe^{2x} **nor** λxe^{2x} can be a particular integral for this equation.

A particular integral has the form kx^2e^{2x}.

c Determine the value of the constant k and find the general solution of the equation.



9 Given that the differential equation $\frac{d^2y}{dt^2} + 4y = 5\cos 3t$ has a particular integral of the form $k\cos 3t$, determine the value of the constant k and find the general solution of the equation.

Find the solution which satisfies the initial conditions that when $t = 0$, $y = 1$ and $\frac{dy}{dt} = 2$.

10 Given that the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{2x}$ has a particular integral of the form $\lambda + \mu x + kxe^{2x}$, determine the values of the constants λ, μ and k and find the general solution of the equation.

11 Find the solution of the differential equation $16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 5y = 5x + 23$ for which $y = 3$ and $\frac{dy}{dx} = 3$ at $x = 0$. Show that $y \approx x + 3$ for large values of x.

12 Find the solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 3\sin 3x - 2\cos 3x$ for which $y = 1$ at $x = 0$ and for which y remains finite for large values of x.

13 Find the general solution of the differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 27\cos t - 6\sin t$.

The equation is used to model water flow in a reservoir. At time t days, the level of the water above a fixed level is x m. When $t = 0$, $x = 3$ and the water level is rising at 6 metres per day.

a Find an expression for x in terms of t.

b Show that after about a week, the difference between the lowest and highest water level is approximately 6 m.

14 **a** Find the general solution of the differential equation

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \ln x, \qquad x > 0,$$

using the substitution $x = e^u$, where u is a function of x.

b Find the equation of the solution curve passing through the point (1, 1) with gradient 1.

15 Solve the equation $\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = \cos^2 x\, e^{\sin x}$, by putting $z = \sin x$, finding the solution for which $y = 1$ and $\frac{dy}{dx} = 3$ at $x = 0$.

Summary of key points

- For the second order differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ **the auxiliary quadratic equation is** $am^2 + bm + c = 0$.

- If the auxiliary equation has **two real distinct roots** α and β (i.e. when $b^2 > 4ac$) , the general solution of the differential equation is $y = Ae^{\alpha x} + Be^{\beta x}$, where A and B are arbitrary constants.

- If the auxiliary equation has **two equal roots** α (i.e. when $b^2 = 4ac$), the general solution of the differential equation is $y = (A + Bx)e^{\alpha x}$, where A and B are arbitrary constants.

- If the auxiliary equation has **two imaginary roots** $\pm i\omega$, (i.e. when $b = 0$ and $4ac > 0$) the general solution of the differential equation is $y = A\cos\omega x + B\sin\omega x$, where A and B are arbitrary constants.

- If the auxiliary equation has **two complex roots** $p \pm iq$ (i.e. when $b^2 < 4ac$), the general solution of the differential equation is $y = e^{px}(A\cos qx + B\sin qx)$, where A and B are arbitrary constants.

- For the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$:
  - First solve $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$.
    The solution, which you obtain, is called the **complementary function**.
  - Then you need to find a solution of the equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$.
    The solution is called **the particular integral**.
  - The general solution of the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, is
    y = **complementary function** + **particular integral**.

- You can use a given substitution to change the variables and transform a second order differential equation into one of the above types of equation, which you can then solve.